INTEGRAL MOMENTS OF AUTOMORPHIC L–FUNCTIONS

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Abstract. We obtain second integral moments of automorphic L–functions on adele groups GL_2 over arbitrary number fields, by a spectral decomposition using the structure and representation theory of adele groups GL_1 and GL_2. This requires reformulation of the notion of *Poincaré series*, replacing the collection of classical Poincaré series over $GL_2(\mathbb{Q})$ or $GL_2(\mathbb{Q}(i))$ with a single, coherent, global object that makes sense over a number field. This is the first expression of integral moments in adele-group terms, distinguishing global and local issues, and allowing uniform application to number fields. When specialized to the field of rational numbers $\mathbb{Q}$, we recover the classical results on moments.

§1. Introduction

For ninety years, mean values of families of automorphic L–functions have played a central role in analytic number theory. In the absence of the Riemann Hypothesis, or of the Grand Riemann Hypothesis referring to general L–functions, suitable mean value results often can substitute. Thus, asymptotics or sharp bounds for integral moments of automorphic L–functions are of intense interest. The study of integral moments was initiated in 1918 by Hardy and Littlewood [Ha-Li], who obtained the asymptotic formula for the second moment of the Riemann zeta-function

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt \ \sim\ T\log T \tag{1.1}$$

Eight years later, Ingham in [I] obtained the fourth moment

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 dt \ \sim\ \frac{1}{2\pi^2}\cdot T(\log T)^4 \tag{1.2}$$

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Since then, many authors have studied moments: for instance, see [At], [HB], [G1], [M1], [J1]. Most results are limited to integral moments of automorphic L–functions for $GL_1(\mathbb{Q})$ and $GL_2(\mathbb{Q})$. No analogue of (1.1) or (1.2) was known over an arbitrary number field. The only previously known results for fields other than $\mathbb{Q}$, are in [M4], [S1], [BM1], [BM2] and [DG2], all over quadratic extensions of $\mathbb{Q}$.

Here we obtain second integral moments of automorphic L–functions on adele groups GL_2 over arbitrary number fields, by a spectral decomposition using the structure and representation theory of adele groups GL_1 and GL_2. This requires reformulation of the notion of *Poincaré series*, replacing the collection of classical Poincaré series over $GL_2(\mathbb{Q})$ or $GL_2(\mathbb{Q}(i))$ with a single global object which makes sense on an adele group over a number field. This is the first expression of integral moments in adele-group-theoretic terms, distinguishing global and local issues, and allowing uniform application to number fields. When specialized to the field of rational numbers $\mathbb{Q}$, we recover the classical results on moments.

More precisely, for f an automorphic form on GL_2 and χ an idele class character of the number field, let $L(s, f\otimes\chi)$ be the twisted L–function attached to f. We obtain asymptotics for averages

$$\sum_{\chi} \int_{-\infty}^{\infty} \left|L\left(\tfrac{1}{2}+it,\, f\otimes\chi\right)\right|^2 M_\chi(t)\, dt \tag{1.3}$$

for suitable smooth weights $M_\chi(t)$. We obtain the asymptotics from the appearance of this sum in *moment expansions* (3.12)

$$\sum_{\chi} \int_{-\infty}^{\infty} L\left(\tfrac{1}{2}+it+v,\, f\otimes\chi\right)\cdot L\left(\tfrac{1}{2}+it,\, \overline{f}\otimes\overline{\chi}\right)\cdot M_\chi(t)\, dt \;=\; \int_{Z_{\mathbb{A}}GL_2(k)\backslash GL_2(\mathbb{A})} \mathrm{Pé}_v\cdot|f|^2 \tag{1.4}$$

The *Poincaré series* $\mathrm{Pé}_v$ has a spectral expansion, which, after an Eisenstein series is removed (see (4.6)), has cuspidal components computed in (4.1) and continuous components computed in (4.10) (there are no residual spectrum components). Thus, using an auxiliary function φ introduced in Section 2, we have *spectral expansions* (see (4.8))

$$\begin{aligned}\int_{Z_{\mathbb{A}}GL_2(k)\backslash GL_2(\mathbb{A})} \mathrm{Pé}_v\cdot|f|^2 \;&=\; \langle E_{v+1},|f|^2\rangle \int_{N_\infty}\varphi_\infty \;+\; \sum_F \bar{\rho}_F\, \mathcal{G}_{F_\infty}(v,w)\, L(v+\tfrac{1}{2},\overline{F})\cdot\langle F,|f|^2\rangle \\ &+ \sum_\chi \frac{\overline{\chi}(\mathfrak{d})}{4\pi i\kappa}\int_{\Re(s)=\frac{1}{2}} \frac{L(v+\bar{s},\overline{\chi})\cdot L(v+1-\bar{s},\chi)}{L(2\bar{s},\overline{\chi}^2)}\,|\mathfrak{d}|^{-(v-\bar{s}+1/2)}\int_{Z_\infty\backslash G_\infty}\varphi_\infty\cdot\overline{W}^{E}_{s,\chi,\infty}\end{aligned} \tag{1.5}$$

where: F is summed over an orthonormal basis for spherical cuspforms on GL_2, E_{v+1} is an Eisenstein series (see (4.6)), $\overline{\rho}_F = \rho_F(1)$ is (in effect) the first Fourier coefficient of the cuspform F, $\mathcal{G}$ is expressed in terms of gamma functions in (4.2) and (4.3), $\mathfrak{d}$ is a differental idele ([W2], page 113, Definition 4) with component 1 at archimedean places, κ is a volume constant (see (4.7)), and W^E is a normalization of Whittaker function attached to Eisenstein series (see Appendix 2).

The sum in (1.3), (1.4), and (1.5) over idele class characters χ is infinite, in general. For general number fields, (1.3) is the correct structure of the second integral moment of GL_2 automorphic L–functions. This was first pointed out by Sarnak in [S1], where such an average was studied over the Gaussian field $\mathbb{Q}(i)$; see also [DG2]. Section 3 shows that this structure reflects Fourier inversion on the idele class group of the field.

Meanwhile, joint work [DGG2] with Goldfeld extends these ideas to produce integral moments for GL_r over number fields, exhibiting explicit kernels Pé again giving identities of the form

$$\text{moment expansion} = \int_{Z_{\mathbb{A}}GL_r(k)\backslash GL_r(\mathbb{A})} \text{Pé} \cdot |f|^2 = \text{spectral expansion}$$

for cuspforms f on GL_r. The moment expansion on the left-hand side is of the form

$$\sum_F \int_{\Re(s)=\frac{1}{2}} |L(s, f \otimes F)|^2 \, M_F(s)\, ds + \ldots$$

summed over F in an orthonormal basis for cuspforms on GL_{r-1}, with corresponding continuous-spectrum terms. The specific choice gives a kernel with a surprisingly simple *spectral expansion*, with only three parts: a leading term, a sum induced from cuspforms on GL_2, and a continuous part again induced from GL_2. In particular, no cuspforms on GL_ℓ with $2 < \ell \le r$ contribute. Since the discussion for GL_r with $r > 2$ depends essentially on the GL_2 case, the latter merits careful attention. Thus we give complete details for GL_2 here. For GL_2 over $\mathbb{Q}$ and square-free level, the sum of moments can be arranged to have a single summand, recovering a classical integral moment

$$\int_{-\infty}^{\infty} \left|L\left(\tfrac{1}{2} + it, f\right)\right|^2 M(t)\, dt$$

As a non-trivial example, consider the case of a cuspform f on GL_3 over $\mathbb{Q}$. We produce a weight function $\Gamma(s, w, f_\infty, F_\infty)$ depending upon complex parameters s and w, and upon the *archimedean* data for both f and cuspforms F on GL_2, such that $\Gamma(s, w, f_\infty, F_\infty)$ has explicit asymptotic behavior similar to that discussed in Section 5 below, and such that the *moment expansion* above becomes

$$\int_{Z_{\mathbb{A}}GL_3(\mathbb{Q})\backslash GL_3(\mathbb{A})} \text{Pé} \cdot |f|^2 = \sum_{F \text{ on } GL_2} \int_{\Re(s)=\frac{1}{2}} |L(s, f \otimes F)|^2 \cdot \Gamma(s, w, f_\infty, F_\infty)\, ds$$

$$+ \sum_{k \in \mathbb{Z}} \int_{\Re(s_1)=\frac{1}{2}} \int_{\Re(s_2)=\frac{1}{2}} |L(s_1, f \otimes E^{(k)}_{1-s_2})|^2 \cdot \Gamma(s_1, w, f_\infty, E^{(k)}_{1-s_2,\infty})\, ds_2\, ds_1$$

where

$$L(s_1, f \otimes E^{(k)}_{1-s_2}) = \frac{L(s_1 - s_2 + \frac{1}{2}, f) \cdot L(s_1 + s_2 - \frac{1}{2}, f)}{\zeta(2 - 2s_2)}$$

Here F runs over an orthonormal basis for all level-one cuspforms on GL_2, without restriction on the right K_∞–type. Similarly, the Eisenstein series $E^{(k)}_s$ run over all level-one Eisenstein series for $GL_2(\mathbb{Q})$ with no restriction on K_∞–type, indexed by k.

The discussion below makes several points clear. First, our sum of moments of twists of L–functions has a natural integral representation on the adele group, of a form insensitive to the underlying number field. Second, the kernel for this adele-group integral arises from a collection of *local* data, wound up into an automorphic form, and the computation proceeds by unwinding. This presentation requires a reformulation of the notion of *Poincaré series*, replacing a weighted sum of classical Poincaré series for $GL_2(\mathbb{Q})$ or $GL_2(\mathbb{Q}(i))$ with a single, coherent, global object that makes sense over a number field. Third, the ramifications of the choices of local data are subtle. In the present treatment we take local data at *finite* primes so as to avoid complications away from archimedean places. Fourth, some subtlety resides in choices of archimedean data. Good's original idea in [G2] for $GL_2(\mathbb{Q})$ can be viewed as a choice of local data for *real* places, which can be improved as in the classical formulation of [DG1]. Similarly, we interpret the treatment of $GL_2(\mathbb{Q}[i])$ in [DG2] as a choice of local data for *complex* primes.

Recipe for spectral identities involving second moments: Spectral identities involving second integral moments of automorphic L-functions and other periods can be produced systematically, as follows. We suppress secondary details, even where non-trivial, writing $\sum\!\!\!\!\!\!\int$ as an *ad hoc* device to indicate integration against a suitable automorphic spectral measure not specified in detail. Thus, for a reductive group Θ over a number field k, with center Z, write the decomposition of functions Ψ in $L^2(Z_{\mathbb{A}}\Theta_k\backslash\Theta_{\mathbb{A}})$ as

$$\Psi \;=\; \sum\!\!\!\!\!\!\!\int_{F \text{ on } \Theta} \langle \Psi, F\rangle_\Theta \;\cdot\; F$$

where the automorphic forms F generate irreducible representations of $\Theta_{\mathbb{A}}$, or nearly so. In the discrete part of this spectral decomposition, the automorphic forms F are genuinely orthonormal. The more continuous part of the decomposition behaves in a less elementary fashion, although most of it admits an explicit description in terms of Eisenstein series. We will not worry about unresolved issues concerning the residual spectrum. Further, we imagine that this L^2 spectral decomposition is extended to suitable *distributions* and whatever non-L^2 functions we need. Regularization of implied integrals is a genuine issue, but not our immediate concern.

Let B be a subgroup of Θ containing Z, and let u be a left $Z_{\mathbb{A}}\Theta_k$-invariant distribution on Θ supported on the image $Z_{\mathbb{A}}B_k\backslash B_{\mathbb{A}}$ inside $Z_{\mathbb{A}}\Theta_k\backslash\Theta_{\mathbb{A}}$. Distributions u involving no derivatives transverse to $Z_{\mathbb{A}}B_k\backslash B_{\mathbb{A}}$ inside $Z_{\mathbb{A}}\Theta_k\backslash\Theta_{\mathbb{A}}$ admit spectral expansions

$$u \;=\; \sum\!\!\!\!\!\!\!\int_{F \text{ on } B} \langle u,\, F\rangle_B \;\cdot\; F$$

Now let Θ be of the form $\Theta = G \times G$, and let Z be the center of G (rather than of Θ). Let H be a k-subgroup of G containing Z. Consider two chains of subgroups inside $G \times G$, pictorially,

$$\begin{array}{ccccc} & & G \times G & & \\ & \nearrow & & \nwarrow & \\ H \times H & & & & G^\Delta \\ & \nwarrow & & \nearrow & \\ & & H^\Delta & & \end{array}$$

where the superscript Δ denotes diagonal copies, and where ascending arrows are inclusions. Consider the left $Z_{\mathbb{A}}G_k \times Z_{\mathbb{A}}G_k$-invariant distribution u on $G \times G$

$$u(f_1 \otimes f_2) \;=\; \int_{Z_{\mathbb{A}}^\Delta H_k^\Delta\backslash H_{\mathbb{A}}^\Delta} f_1 \otimes f_2 \;=\; \int_{Z_{\mathbb{A}} H_k\backslash H_{\mathbb{A}}} f_1 \cdot f_2$$

The spectral expansion of u along $H \times H$ has a special *diagonal* property, namely

$$u \;=\; \sum\!\!\!\!\!\!\!\int_{F_1, F_2 \text{ on } H} \langle u,\ F_1 \otimes F_2 \rangle_{H \times H} \ \cdot F_1 \otimes F_2 \;=\; \sum\!\!\!\!\!\!\!\int_{F \text{ on } H} \overline{F} \otimes F$$

The diagonal feature of this expansion is completely analogous to the fact that the Fourier series expansion of the distribution that integrates along the diagonal circle in a product of two circles has Fourier coefficients only along the diagonal. Then

$$u(f \otimes \overline{f}) \;=\; \sum\!\!\!\!\!\!\!\int_{F \text{ on } H} \langle f \otimes \overline{f}, F \otimes \overline{F} \rangle_{H \times H} \;=\; \sum\!\!\!\!\!\!\!\int_{F \text{ on } H} |\langle f,\ F \rangle_H|^2$$

The *positivity* of the summands is a virtue of this relation.

On the other hand, the spectral expansion of u along G^Δ is

$$u \;=\; \sum\!\!\!\!\!\!\!\int_{F \text{ on } G^\Delta} \langle u,\ \overline{F} \rangle_{G^\Delta} \cdot \overline{F} \;=\; \sum\!\!\!\!\!\!\!\int_{F \text{ on } G^\Delta} F_H \cdot \overline{F}$$

where $F_H = \langle u, \overline{F} \rangle$ is the *period* of F along H. Thus,

$$u(f \otimes \overline{f}) \;=\; \sum\!\!\!\!\!\!\!\int_{F \text{ on } G^\Delta} F_H \cdot \langle f \otimes \overline{f},\ \overline{F} \rangle_{G^\Delta} \;=\; \sum\!\!\!\!\!\!\!\int_{F \text{ on } G} F_H \ \cdot\ \langle F,\ |f|^2 \rangle_G$$

The *period non-vanishing* condition $F_H \neq 0$, for F to appear in the expansion, is non-trivial.

Equating these two expansions gives

$$\sum\!\!\!\!\!\!\!\int_{F \text{ on } H} |\langle f, F \rangle_H|^2 \;=\; u(f \otimes \overline{f}) \;=\; \sum\!\!\!\!\!\!\!\int_{F \text{ on } G} F_H \ \cdot\ \langle F,\ |f|^2 \rangle_G$$

Diagrammatically, this is

$$\begin{array}{ccccccc}
\text{(moment side)} & & & & & & \text{(spectral side)} \\
\displaystyle\sum\!\!\!\!\!\!\!\int_{F \text{ on } H} |\langle f, F \rangle_H|^2 & \longleftarrow & & \begin{array}{c} f \otimes \overline{f} \\ G \times G \end{array} & & \longrightarrow & \displaystyle\sum\!\!\!\!\!\!\!\int_{F \text{ on } G} F_H \ \cdot\ \langle F,\ |f|^2 \rangle_G \\
\uparrow & & \nearrow & & \nwarrow & & \uparrow \\
\displaystyle\sum\!\!\!\!\!\!\!\int_{F \text{ on } H} F \otimes \overline{F} & & H \times H & & G^\Delta & & \displaystyle\sum\!\!\!\!\!\!\!\int_{F \text{ on } G^\Delta} F_H \cdot F \\
 & & & \nwarrow \quad\quad \nearrow & & & \\
 & & & \begin{array}{c} H^\Delta \\ u \sim 1 \end{array} & & &
\end{array}$$

The evaluation along $H \times H$ is especially interesting when H is an *Euler-Gelfand* subgroup of G, in the sense that restrictions from G_v to H_v of (possibly a restricted class of) irreducibles on G_v are multiplicity-free. This makes $\langle f, F \rangle_H$ tend to have an Euler product, possibly including a period. Thus, the left-hand (*moment*) side of the spectral identity is a sum (and integral) of *second* *integral moments* of L-functions. By contrast, the right-hand (*spectral*) side inevitably involves integrals of three eigenfunctions, for automorphic forms F with non-vanishing periods F_H.

The exponential decay of the archimedean contributions in this identity make the sums converge too well, so u must be *deformed* to extract more information. Specifically, call any deformation of the initial distribution u to a (*classical*, rather than *generalized*) function on $Z_{\mathbb{A}}^{\Delta} G_k^{\Delta} \backslash G_{\mathbb{A}}^{\Delta}$ a *Poincaré series*. A simple family of Poincaré series is constructed by winding up *local* data, created by *local* deformations, as follows. Let $\varphi = \bigotimes_\nu \varphi_\nu$ where φ_ν is a left H_ν-invariant function on the ν-adic points G_ν of G. Form a Poincaré series

$$\text{Pé}_\varphi(g) \;=\; \sum_{\gamma \in H_k \backslash G_k} \varphi(\gamma \cdot g)$$

To have φ_ν be a classical function on G_ν at non-archimedean places ν, we have the option to make an extremely simple choice

$$\varphi_\nu(g) \;=\; \begin{cases} 1 & (\text{for } g \in H_\nu \cdot K_\nu) \\ \\ 0 & (\text{otherwise}) \end{cases}$$

where K_ν is (maximal) compact in G_ν. Archimedean places do not usually allow such trivial local deformations.

In this recipe, taking $H = GL_1$ and $G = GL_2$ gives the L-functions $L(s, f \otimes \chi)$ considered in this paper, with f on GL_2 and grossencharacters χ. Extending the $GL_2 \times GL_1$ case, we obtain convolution L-functions $L(s, f \otimes F)$ on $GL_n \times GL_{n-1}$ by taking $H = GL_{n-1}$ and $G = GL_n$. On the spectral side are *standard* L-functions attached to F on GL_{n-1}, along with triple integrals of eigenfunctions. Due to vanishing of GL_{n-1} periods, the spectral side involves no cuspidal data on GL_r for $r > 2$. Rankin-Selberg convolutions for $GL_n \times GL_n$ arise by taking $H = GL_n^{\Delta}$ and $G = GL_n \times GL_n$. In the smallest case $n = 2$, the subgroup G^{Δ} is also Euler-Gelfand in $G \times G$. Rankin-Selberg L-functions given by *doubling* arise as follows. Let Φ be a form (orthogonal, symplectic, or hermitian) on a suitable vectorspace, and $U(\Phi)$ its isometry group. Let $H = U(\Phi) \times U(-\Phi)$ and $G = U(\Phi \oplus -\Phi)$ and take Siegel-type Eisenstein series on G. The period non-vanishing condition on the spectral side is again a stringent condition in this example. Triple-product L-functions appear in at least two ways. First, one may take $G = GSp_3$ and $H = (GL_2 \times GL_2 \times GL_2)^{\natural}$, where the $\natural$ means to take the subgroup where the three determinants are equal. Take Siegel-type Eisenstein series on G. A smaller family appears by taking $G = GSp_2$ and $H = (GL_2 \times GL_2)^{\natural}$, with an Eisenstein series on G with cuspidal data.

Of course, the general idea of exploiting decompositions of automorphic forms or representations along subgroups is a main technical device in the theory of automorphic forms, and is several decades old. Indeed, restriction and decomposition along simple inclusions $A \subset B$ for Euler-Gelfand subgroups A of groups B have been studied for decades. By contrast, consideration of larger configurations of subgroups and iterated spectral decomposition along them is much less clichéd, due in part to technical complications, and due also to a paucity of suggestive examples. For us, the chief desired effect is a *positivity* property of a sum-and-integral of Euler products on one side of the identity, attached to an Euler-Gelfand subgroup H of G. Our *recipe* above accomplishes this for arbitrary inclusions $H^{\Delta} \subset H \times H$ and $G^{\Delta} \subset G \times G$, *regardless* of whether these diagonals are Euler-Gelfand subgroups. It may be that Reznikov's notable recent work [R] is the only other current example of systematic use of iterated spectral decompositions along larger configurations of subgroups. Given the scope of possibilities in considering the spectral theory of large configurations, it is not surprising that [R] offers a somewhat different recipe, under somewhat different hypotheses, achieving somewhat different ends than we have indicated here. Leaving a more detailed comparison and systematization to the interested reader, we take the viewpoint that, in any case, our present discussion and that of [R] offer persuasive evidence for the utility of iterated spectral decompositions in large configurations of subgroups.

The structure of the paper is as follows. In Section 2, integral kernels we call *Poincaré series* are described in terms of local data, reformulating classical examples in a form applicable to GL_r over a number field. In Section 3, the integral of the Poincaré series against $|f|^2$ for a cuspform f on GL_2 is unwound and expanded, yielding a sum of weighted moment integrals of L–functions $L(s, f \otimes \chi)$ of twists of f by idele class characters χ. In Section 4, the spectral decomposition of the Poincaré series is exhibited: the leading term is an Eisenstein series, and there are cuspidal and continuous-spectrum parts with coefficients which are values of L–functions. In Section 5, an asymptotic formula is derived for these integral moments. We note there that the *length* of the averages involved is amenable to subsequent applications to convexity breaking in the t–aspect. The first appendix discusses convergence of the Poincaré series in detail, proving pointwise convergence and L^2 convergence. The second appendix computes integral transforms necessary to understand some details in the spectral expansion.

For the most immediate applications, such as subconvexity, refined choices of archimedean data must be combined with the generalization [HR] of [HL], invoking [Ba], as well as an extension of [S2] (or [BR]) to number fields. However, for now, we content ourselves with laying the groundwork for applications and extensions. In subsequent papers we will address the extension of the identity to GL_r, and consider convexity breaking in the t–aspect.

§2. Poincaré series

Let k be a number field, $G = GL_r$ over k, and define standard subgroups:

$$P = P^{r-1,1} = \left\{ \begin{pmatrix} (r-1)\text{-by-}(r-1) & * \\ 0 & 1\text{-by-}1 \end{pmatrix} \right\}$$

$$U = \left\{ \begin{pmatrix} I_{r-1} & * \\ 0 & 1 \end{pmatrix} \right\} \qquad H = \left\{ \begin{pmatrix} (r-1)\text{-by-}(r-1) & 0 \\ 0 & 1 \end{pmatrix} \right\} \qquad Z = \text{center of } G$$

Let K_ν denote the standard maximal compact in the k_ν–valued points G_ν of G.

The **Poincaré series** $\text{Pé}(g)$ is of the form

$$\text{Pé}(g) \;=\; \sum_{\gamma \in Z_k H_k \backslash G_k} \varphi(\gamma g) \qquad (g \in G_{\mathbb{A}}) \tag{2.1}$$

for suitable functions φ on $G_{\mathbb{A}}$ described as follows. For $v \in \mathbb{C}$, let

$$\varphi \;=\; \bigotimes_\nu \varphi_\nu \tag{2.2}$$

where for ν *finite*

$$\varphi_\nu(g) \;=\; \begin{cases} \left|(\det A)/d^{r-1}\right|_\nu^v & \text{for } g = mk \text{ with } m = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in Z_\nu H_\nu \text{ and } k \in K_\nu \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

and for ν *archimedean* require right K_ν–invariance and left equivariance

$$\varphi_\nu(mg) = \left|\frac{\det A}{d^{r-1}}\right|_\nu^v \cdot \varphi_\nu(g) \qquad \left(\text{for } g \in G_\nu \text{ and } m = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in Z_\nu H_\nu\right) \tag{2.4}$$

Thus, for $\nu|\infty$, the further data determining φ_ν consists of its values on U_ν. The simplest useful choice is

$$\varphi_\nu \begin{pmatrix} I_{r-1} & x \\ 0 & 1 \end{pmatrix} = \left(1 + |x_1|^2 + \cdots + |x_{r-1}|^2\right)^{-d_\nu(r-1)w_\nu/2} \qquad \left(x = \begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix} \text{ and } w_\nu \in \mathbb{C}\right) \tag{2.5}$$

with $d_\nu = [k_\nu : \mathbb{R}]$. Here the norm $|x_1|^2 + \cdots + |x_{r-1}|^2$ is invariant under K_ν, that is, $|\cdot|$ is the usual absolute value on $\mathbb{R}$ or $\mathbb{C}$. Note that by the product formula φ is left $Z_{\mathbb{A}}H_k$–invariant.

Proposition 2.6. (Apocryphal) *With the specific choice (2.5) of $\varphi_\infty = \otimes_{\nu|\infty}\varphi_\nu$, the series (2.1) defining Pé(g) converges absolutely and locally uniformly for $\Re(v) > 1$ and $\Re(w_\nu) > 1$ for all $\nu|\infty$.*

Proof: In fact, the argument applies to a much broader class of archimedean data. For a complete argument when $r = 2$, and $w_\nu = w$ for all $\nu|\infty$, see Appendix 1. □

We can give a broader and more robust, though somewhat weaker, result, as follows. Again, for simplicity, take $r = 2$. Given φ_∞, for x in $k_\infty = \prod_{\nu|\infty} k_\nu$, let

$$\Phi_\infty(x) \;=\; \varphi_\infty \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

For $0 < \ell \in \mathbb{Z}$, let Ω_ℓ be the collection of φ_∞ such that the associated Φ_∞ is absolutely integrable, and such that the Fourier transform $\widehat{\Phi}_\infty$ along k_∞ satisfies the bound

$$\widehat{\Phi}_\infty(x) \;\ll\; \prod_{\nu|\infty} (1 + |x|_\nu)^{-\ell}$$

For example, for φ_∞ to be in Ω_ℓ it suffices that Φ_∞ is $\ell + 1$ times continuously differentiable, with each derivative absolutely integrable. The simple explicit choice of φ_∞ above lies in Ω_ℓ for *every* $\ell > 0$, when $\Re(w_\nu) > 1$ and $\Re(v) > 1$ for convergence.

Theorem 2.7. (Apocryphal) *Take $r = 2$, $\Re(v) > 3$, $\ell > \Re(v) + 5$, and $\varphi_\infty \in \Omega_\ell$. The series defining Pé$(g)$ converges absolutely and locally uniformly in both g and v. Furthermore, up to an Eisenstein series, the Poincaré series is square integrable on $Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}$.*

Proof: See Appendix 1. □

We do not claim that the lower bounds are the best possible, but only that they arise as artifacts in the natural argument given in Appendix 1. The precise Eisenstein series to be subtracted from the Poincaré series to make the latter square-integrable will be discussed in Section 4 (see formula 4.6). For our special choice (2.5) of archimedean data, both these convergence results apply with $\Re(w_\nu) > 1$ for $\nu|\infty$ and $\Re(v)$ large.

A monomial vector φ as in (2.2) described by (2.3) and (2.4) will be called *admissible* when $\varphi_\infty \in \Omega_\ell$, with $\Re(v) > 3$ and $\ell > \Re(v) + 5$.

§3. Unwinding to an Euler product

Unlike classical contexts, where the Euler factorization of a Dirichlet series is visible only at the end, the present construction presents us with an Euler product almost immediately. From now on, take $r = 2$, so $G = GL_2$ over a number field k, and

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \qquad N = U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \qquad H = \left\{ \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \right\} \qquad M = ZH = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$$

For a place ν of k, let K_ν be the standard maximal compact subgroup. That is, at finite places $K_\nu = GL_2(\mathfrak{o}_\nu)$, at real places $K_\nu = O(2)$, and at complex places $K_\nu = U(2)$.

Using the Poincaré series defined by (2.1), we unwind a corresponding global integral and express it as an inverse Mellin transform of an Euler product. This produces a sum over Hecke characters of weighted integrals of corresponding L–functions over the critical line. Recall the definition

$$\text{Pé}(g) \;=\; \sum_{\gamma \in M_k \backslash G_k} \varphi(\gamma g) \qquad (g \in G_{\mathbb{A}}) \tag{3.1}$$

where the *monomial* vector

$$\varphi = \bigotimes_\nu \varphi_\nu$$

is

$$\varphi_\nu(g) \;=\; \begin{cases} \chi_{0,\nu}(m) & \text{for } g = mk,\ m \in M_\nu \text{ and } k \in K_\nu \\ 0 & \text{for } g \notin M_\nu \cdot K_\nu \end{cases} \qquad (\text{for } \nu \text{ finite}) \tag{3.2}$$

and for ν infinite, we do not entirely specify φ_ν, only requiring the left equivariance

$$\varphi_\nu(mnk) = \chi_{0,\nu}(m) \cdot \varphi_\nu(n) \qquad (\text{for } \nu \text{ infinite, } m \in M_\nu,\ n \in N_\nu \text{ and } k \in K_\nu) \tag{3.3}$$

Here, $\chi_{0,\nu}$ is the character of M_ν given by

$$\chi_{0,\nu}(m) = \left|\frac{a}{d}\right|_\nu^v \qquad \left(m = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_\nu,\ v \in \mathbb{C} \right) \tag{3.4}$$

Then $\chi_0 = \bigotimes_\nu \chi_{0,\nu}$ is M_k–invariant, and φ has trivial central character and is left $M_{\mathbb{A}}$–equivariant by χ_0. Note that for ν infinite, the assumptions imply that

$$x \longrightarrow \varphi_\nu \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a function of $|x|$ only.

Let f_1 and f_2 be cuspforms on $G_{\mathbb{A}}$. Eventually we will take $f_1 = f_2$, but for now merely require the following compatibilities. Suppose that the representations of $G_{\mathbb{A}}$ generated by f_1 and f_2 are *irreducible*, with the same central character. At all ν, require that f_1 and f_2 have the same right K_ν–type, that this K_ν–type is *irreducible*, and that f_1 and f_2 correspond to the same vector in the K–type, up to scalar multiples. Schur's lemma assures that this makes sense, insofar as there are no non-scalar automorphisms. Last, require that each f_i is a special vector locally everywhere in the representation it generates, in the following sense. Let

$$f_i(g) \;=\; \sum_{\xi \in Z_k \backslash M_k} W_{f_i}(\xi g) \tag{3.5}$$

be the Fourier expansion of f_i, and let

$$W_{f_i} = \bigotimes_{\nu \leq \infty} W_{f_i, \nu}$$

be the factorization of the Whittaker function W_{f_i} into local data. By [JL], we may require that for all $\nu < \infty$ the Hecke–type local integrals

$$\int\limits_{a \in k_\nu^\times} W_{f_i,\,\nu} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|_\nu^{s-\frac{1}{2}} \, da$$

differ by at most an exponential function from the correct local L–factors for the representation generated by f_i.

Suppressing some details in the notation, the integral under consideration is

$$I(\chi_0) = \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \text{Pé}(g)\, f_1(g)\, \bar{f}_2(g)\, dg \tag{3.6}$$

For χ_0 (and archimedean data) in the range of absolute convergence, from the definition of the Poincaré series, the integral unwinds to

$$\int_{Z_{\mathbb{A}} M_k \backslash G_{\mathbb{A}}} \varphi(g)\, f_1(g)\, \bar{f}_2(g)\, dg$$

Using the Fourier expansion

$$f_1(g) \;=\; \sum_{\xi \in Z_k \backslash M_k} W_{f_1}(\xi\, g)$$

this further unwinds to

$$\int_{Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} \varphi(g)\, W_{f_1}(g)\, \bar{f}_2(g)\, dg \tag{3.7}$$

Let $\mathbb{J}$ be the ideles, let C be the idele class group $\mathbb{J}/k^\times = GL_1(\mathbb{A})/GL_1(k)$, and let $\widehat{C}$ be the dual of C. By Fujisaki's Lemma (see Weil [W1], page 32, Lemma 3.1.1), the idele class group C is a product of a copy of $(0, \infty)$ and a compact group C_0. By Pontryagin duality, $\widehat{C} \approx \mathbb{R} \times \widehat{C}_0$ with $\widehat{C}_0$ discrete. For any compact open subgroup U_{fin} of the finite-prime part in C_0, the dual of C_0/U_{fin} is finitely generated with rank $[k : \mathbb{Q}] - 1$. On C the spectral decomposition (Fourier-Mellin inversion) for a suitable function F is

$$\begin{aligned} F(x) \;&=\; \int_{\widehat{C}} \int_C F(y) \chi(y)\, dy\; \chi^{-1}(x)\, d\chi \\ &=\; \sum_{\chi' \in \widehat{C}_0} \frac{1}{2\pi i} \int\limits_{\Re(s) = \sigma} \int_C F(y) \chi'(y) |y|^s \, dy\; {\chi'}^{-1}(x) |x|^{-s}\, ds \end{aligned} \tag{3.8}$$

for a suitable Haar measure on C.

For ν infinite and $s \in \mathbb{C}$, let

$$\begin{aligned} \mathcal{K}_\nu(s,\, \chi_{0,\nu},\, \chi_\nu) \;=\; & \int_{Z_\nu\backslash M_\nu N_\nu} \int_{Z_\nu\backslash M_\nu} \varphi_\nu(m_\nu n_\nu) W_{f_1,\nu}(m_\nu n_\nu) \\ & \cdot \overline{W}_{f_2,\nu}(m'_\nu n_\nu)\, \chi_\nu(m'_\nu)\, |m'_\nu|_\nu^{s-\frac12}\, \chi_\nu(m_\nu)^{-1}\, |m_\nu|_\nu^{\frac12 - s}\, dm'_\nu\, dn_\nu\, dm_\nu \end{aligned} \tag{3.9}$$

and set

$$\mathcal{K}_\infty(s,\, \chi_0,\, \chi) \;=\; \prod_{\nu|\infty} \mathcal{K}_\nu(s,\, \chi_{0,\nu},\, \chi_\nu) \tag{3.10}$$

Here $\chi_0 = \bigotimes_\nu \chi_{0,\nu}$ is the character defining the monomial vector φ, and $\chi = \bigotimes_\nu \chi_\nu \in \widehat{C}_0$. For *admissible* φ, the integral (3.9) converges absolutely for $\Re(s)$ sufficiently large. We are especially interested in the choice

$$\varphi_\nu(n) \;=\; \begin{cases} \left(1+x^2\right)^{-\frac{w}{2}} & \text{for } \nu|\infty \text{ real, and } n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N_\nu \\[2ex] \left(1+(x\bar{x})\right)^{-w} & \text{for } \nu|\infty \text{ complex, and } n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N_\nu \end{cases} \qquad (v,\, w \in \mathbb{C}) \tag{3.11}$$

The monomial vector φ generated by this choice is admissible for $\Re(w) > 1$ and $\Re(v) > 3$ (see Appendix 1), and in Section 5 will yield an asymptotic formula for the GL_2 integral moment over the number field k. The main result of this section is

Theorem 3.12. *For φ an admissible monomial vector as above, for suitable $\sigma > 0$,*

$$\int_{Z_{\mathbb{A}} G_k\backslash G_{\mathbb{A}}} P\acute{e} \cdot f_1 \cdot \overline{f}_2 \;=\; I(\chi_0) \;=\; \sum_{\chi\in\widehat{C}_0} \frac{1}{2\pi i} \int_{\Re(s)=\sigma} L(1-s+v,\, f_1\otimes\overline{\chi}) \cdot L(s, \bar{f}_2 \otimes \chi)\, \mathcal{K}_\infty(s,\, \chi_0,\, \chi)\, ds$$

For a finite set of places S including archimedean places, all absolutely ramified primes, and all finite bad places for f_1 and f_2, the sum is over a set $\widehat{C}_{0,S}$ of characters unramified outside S, with bounded ramification at finite places, depending only upon f_1 and f_2.

Proof: Let $\mathbb{J}$ be the ideles of k. Via the identification

$$H_k\backslash H_{\mathbb{A}} \;=\; \left\{ \begin{pmatrix} a' & 0 \\ 0 & 1 \end{pmatrix} : a' \in \mathbb{J}/k^\times \right\} \approx C$$

for a Hecke character χ and for $a \in \mathbb{J}$, write

$$\chi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \;=\; \chi(a)$$

Applying (3.8) to $\bar{f}_2$ and using the Fourier expansion

$$f_2(g) \;=\; \sum_{\xi\in Z_k\backslash M_k} W_{f_2}(\xi\, g)$$

the integral (3.7) is

$$\int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\, W_{f_1}(g) \left(\int_{\widehat{C}} \int_{H_k\backslash H_{\mathbb{A}}} \bar{f}_2(m'g)\, \chi(m')\, dm'\, d\chi \right) dg$$

$$= \int_{\widehat{C}} \left(\int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\, W_{f_1}(g) \int_{H_k\backslash H_{\mathbb{A}}} \sum_{\xi\in H_k} \overline{W}_{f_2}(\xi m'g)\, \chi(m')\, dm'\, dg \right) d\chi$$

$$= \int_{\widehat{C}} \left(\int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\, W_{f_1}(g) \int_{H_{\mathbb{A}}} \overline{W}_{f_2}(m'g)\, \chi(m')\, dm'\, dg \right) d\chi \qquad \text{(identifying } H_{\mathbb{A}} = \mathbb{J}\text{)}$$

The interchange of order of integration is justified by the absolute convergence of the outer two integrals, from the rapid decay of cuspforms along the split torus.

For fixed f_1 and f_2, the finite-prime ramification of the characters $\chi \in \widehat{C}$ is bounded, so there are only finitely many bad finite primes for all the χ which appear. In particular, all the characters χ which appear are unramified outside S and with bounded ramification, depending only on f_1 and f_2, at finite places in S. Thus, for $\nu \in S$ finite, there exists a compact open subgroup U_ν of $\mathfrak{o}_\nu^\times$ such that the kernel of the ν^{th} component χ_ν of χ contains U_ν for all characters χ which appear.

Since f_1 and f_2 generate irreducibles locally everywhere, by uniqueness of Whittaker models [JL], the Whittaker functions W_{f_i} factor

$$W_{f_i}(\{g_\nu : \nu \le \infty\}) = \Pi_\nu W_{f_i,\nu}(g_\nu)$$

Therefore, the inner integral over $Z_{\mathbb{A}}\backslash G_{\mathbb{A}}$ and $\mathbb{J}$ factors over primes, and

$$I(\chi_0) = \int_{\widehat{C}} \Pi_\nu \left(\int_{Z_\nu\backslash G_\nu} \int_{H_\nu} \varphi_\nu(g_\nu)\, W_{f_1,\nu}(g_\nu)\, \overline{W}_{f_2,\nu}(m'_\nu g_\nu)\, \chi_\nu(m'_\nu)\, dm'_\nu\, dg_\nu \right) d\chi$$

Suppress the reference to the place ν to write the ν^{th} local integral more cleanly, as

$$\int_{Z\backslash G} \int_H \varphi(g)\, W_{f_1}(g)\, \overline{W}_{f_2}(m'g)\, \chi(m')\, dm'\, dg$$

Take ν *finite* such that both f_1 and f_2 are right K_ν–invariant. With a ν–adic Iwasawa decomposition $g = mnk$ with $m \in M$, $n \in N$, and $k \in K$, the Haar measure is $d(mnk) = dm\, dn\, dk$ with Haar measures on the factors. The integral becomes

$$\int_{Z\backslash MN} \int_H \varphi(mn)\, W_{f_1}(mn)\, \overline{W}_{f_2}(m'mn)\, \chi(m')\, dm'\, dn\, dm$$

Use representatives H for $Z\backslash M$. To symmetrize the integral, replace m' by $m'm^{-1}$ to obtain

$$\int_{HN} \int_H \varphi(mn)\, W_{f_1}(mn)\, \overline{W}_{f_2}(m'n)\, \chi(m')\, \chi(m)^{-1}\, dm'\, dn\, dm$$

The Whittaker functions W_{f_i} have left N–equivariance

$$W_{f_i}(ng) = \psi(n)\, W_{f_i}(g) \quad \text{(fixed non-trivial } \psi\text{)}$$

so

$$W_{f_i}(mn) = W_{f_i}(mnm^{-1}\,m) = \psi(mnm^{-1})\,W_{f_i}(m)$$

Thus, letting

$$X(m, m') = \int_N \varphi(n)\,\psi(mnm^{-1})\,\overline{\psi}(m'nm'^{-1})\,dn$$

the local integral is

$$\int_H \int_H \chi_0(m)\,W_{f_1}(m)\,\overline{W}_{f_2}(m')\,\chi(m')\,\chi^{-1}(m)\,X(m, m')\,dm'\,dm$$

We claim that for m and m' in the supports of the Whittaker functions, the inner integral $X(m, m')$ is constant, independent of m, m', and it is 1 for almost all finite primes. First, $\varphi(mn)$ is 0, unless $n \in M \cdot K \cap N$, that is, unless $n \in N \cap K$. On the other hand,

$$\psi(mnm^{-1}) \cdot W_{f_1}(mk) = \psi(mnm^{-1}) \cdot W_{f_1}(m) = W_{f_1}(mn) = W_{f_1}(m) \qquad (\text{for } n \in N \cap K)$$

Thus, for $W_{f_1}(m) \neq 0$, necessarily $\psi(mnm^{-1}) = 1$. A similar discussion applies to W_{f_2}. So, up to normalization, the inner integral is 1 for m, m' in the supports of W_{f_1} and W_{f_2}. Then

$$\begin{aligned} &\int_H \int_H \chi_0(m)\,W_{f_1}(m)\,\overline{W}_{f_2}(m')\,\chi(m')\,\chi^{-1}(m)\,dm\,dm' \\ &= \int_H (\chi_0 \cdot \chi^{-1})(m)\,W_{f_1}(m)\,dm \cdot \int_H \chi(m')\,\overline{W}_{f_2}(m')\,dm' \\ &= L_\nu(\chi_{0,\nu} \cdot \chi_\nu^{-1} |\cdot|_\nu^{1/2},\, f_1) \cdot L_\nu(\chi_\nu |\cdot|_\nu^{1/2},\, \bar{f}_2) \end{aligned}$$

i.e., the product of local factors of the standard L–functions in the theorem, up to exponential functions at finitely many finite primes, by our assumptions on f_1 and f_2.

For non-trivial right K–type σ, the argument is similar but a little more complicated. The key point is that the inner integral over N (as above) should not depend on mk and $m'k$, for mk and $m'k$ in the support of the Whittaker functions. Changing conventions for a moment, look at V_σ–valued Whittaker functions, and consider any W in the ν^{th} Whittaker space for f_i having right K–isotype σ. Thus,

$$W(gk) = \sigma(k) \cdot W(g) \qquad (\text{for } g \in G \text{ and } k \in K)$$

For $\varphi(mn) \neq 0$, again $n \in N \cap K$. Then

$$\sigma(k) \cdot \psi(mnm^{-1}) \cdot W(m) = W(mnk) = \sigma(k) \cdot W(mn) = \sigma(k) \cdot \sigma(n) \cdot W(m)$$

where in the last expression n comes out on the right by the right σ–equivariance of W. For m in the support of W, $\sigma(n)$ acts by the scalar $\psi(mnm^{-1})$ on $W(mk)$, for all $k \in K$. Thus, $\sigma(n)$ is scalar on that copy of V_σ. At the same time, this scalar is $\sigma(n)$, so is independent of m if $W(m) \neq 0$. Thus, except for a common integral over K, the local integral falls into two pieces, each yielding the local factor of the L–function. From Schur orthogonality, the common integral over K is a constant, non-zero since the two vectors are collinear in the K–type. □

At this point the archimedean local factors of the Euler product are not entirely specified. The option to vary the choices is useful in applications.

§4. Spectral decomposition of Poincaré series

Now spectrally decompose the Poincaré Pé series defined in (3.1). Throughout this section, assume that φ is *admissible* in the sense given at the end of Section 2. We shall see that $\mathrm{Pé}(g)$ is not generally square-integrable. However, by choosing the archimedean part of the monomial vector φ to have enough decay, and by subtracting an obvious Eisenstein series, the remainder is in L^2 and has sufficient decay so that its integrals against Eisenstein series converge absolutely, by explicit computation. In particular, when the archimedean data is specialized to (3.11), the Poincaré series $\mathrm{Pé}(g)$ has meromorphic continuation in the variables v and w: this follows from the spectral decomposition, from the meromorphic continuation of the spectral fragments, and from standard estimates on the aggregate. See [DG1], [DG2] when $k=\mathbb{Q}$, $\mathbb{Q}(i)$.

Let k be a number field, $G = GL_2$ over k, and ω a unitary character of $Z_k\backslash Z_{\mathbb{A}}$. From [GGPS], [GJ], or [Go1] and [Go2], recall the decomposition

$$L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}},\,\omega) \;=\; L^2_{\mathrm{cusp}}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}},\,\omega) \,\oplus\, L^2_{\mathrm{cusp}}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}},\,\omega)^{\perp}$$

where $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}},\,\omega)$ is L^2 functions with central character ω, and where $L^2_{\mathrm{cusp}}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}},\,\omega)$ is L^2 cuspforms with central character ω. The orthogonal complement to cuspforms consists of one-dimensional representations (the residual spectrum here) and integrals of Eisenstein series:

$$\begin{aligned} L^2_{\mathrm{cusp}}(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}},\,\omega)^{\perp} \;\approx\; & \{1-\text{dimensional representations}\} \\ & \oplus \int^{\oplus}_{(GL_1(k)\backslash GL_1(\mathbb{A}))^{\widehat{}}} \bigotimes_{\nu} \mathrm{Ind}_{P_\nu}^{G_\nu}(\chi_\nu\,\delta_\nu^{1/2})\,d\chi \end{aligned}$$

where δ is the modular function on $P_{\mathbb{A}}$, and the isomorphism is via Eisenstein series. Using this, with central character ω trivial for our Poincaré series, explicitly decompose the Poincaré series as

$$\mathrm{Pé} \;=\; \text{Eisenstein series} \;+\; \text{discrete part} \;+\; \text{continuous part}$$

The projection to cuspforms is straightforward componentwise:

Proposition 4.1. *Let f be a cuspform on $G_{\mathbb{A}}$ generating a spherical representation locally everywhere, and suppose f corresponds to a spherical vector everywhere locally. In the region of absolute convergence of the Poincaré series $\mathrm{Pé}(g)$, the integral*

$$\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \bar f(g)\,\mathrm{Pé}(g)\,dg$$

is an Euler product. At finite ν, the corresponding local factors are $L_\nu(\chi_{0,\nu}\,|\cdot|_\nu^{1/2},\,\bar f\,)$, up to multiplicative constants depending on the set of absolutely ramified primes in k.

Of course, the spectral decomposition of a right $K_{\mathbb{A}}$–invariant automorphic form only involves everywhere locally *spherical* cuspforms. Thus, the following computations can ignore holomorphic discrete series and non-spherical principal series representations.

Proof: The computation uses the same facts as did the Euler factorization in the previous section. From the Fourier expansion

$$f(g) \;=\; \sum_{\xi\in Z_k\backslash M_k} W(\xi g)$$

unwind

$$\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \bar{f}(g)\,\text{Pé}(g)\,dg \;=\; \int_{Z_{\mathbb{A}}M_k\backslash G_{\mathbb{A}}} \sum_{\xi} \overline{W}(\xi g)\,\varphi(g)\,dg \;=\; \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \overline{W}(g)\,\varphi(g)\,dg$$

$$= \prod_{\nu} \left(\int_{Z_\nu\backslash G_\nu} \overline{W}_\nu(g_\nu)\,\varphi_\nu(g_\nu)\,dg_\nu \right)$$

where the local Whittaker functions at finite places are normalized as in [JL] to give the *correct* local L–factors.

At finite ν, suppressing the subscript ν, the integrand in the ν^{th} local integral is right K_ν–invariant, so we can integrate over $Z\backslash MN \approx HN$ with left Haar measure. The ν^{th} Euler factor is

$$\int_H \int_N \overline{W}(mn)\,\varphi(mn)\;dn\,dm = \int_H \int_N \overline{\psi}(mnm^{-1})\,\overline{W}(m)\,\chi_0(m)\,\varphi(n)\,dn\,dm$$

for all finite primes ν. The integral over n is

$$\int_N \overline{\psi}(mnm^{-1})\,\varphi(n)\,dn$$

For $\varphi(n)$ to be non-zero requires n to lie in $M\cdot K$, which further requires, as before, that $n \in N\cap K$. Again, $W(m) = 0$ unless

$$m(N\cap K)m^{-1} \subset N\cap K$$

The character ψ is trivial on $N\cap K$. Thus, the integral over N is really the integral of 1 over $N\cap K$. Thus, at finite primes ν, the local factor is

$$\int_H \overline{W}(m)\,\chi_0(m)\,dm \;=\; L_\nu(\tfrac{1}{2}+v,,\,\bar{f}\,) \qquad \square$$

Let φ be given by (3.11). Take $\Re(v) > 1$ and $\Re(w) > 1$ to ensure absolute convergence of $\text{Pé}(g)$, by Proposition 2.6. The local integral in Proposition 4.1 at infinite ν is

$$\int_{Z_\nu\backslash G_\nu} \overline{W}_\nu(g_\nu)\,\varphi_\nu(g_\nu)\,dg_\nu \;=\; \mathcal{G}_\nu(\tfrac{1}{2}+i\bar{\mu}_{f,\nu};v,w)$$

where, up to a constant, at real places ν

$$\mathcal{G}_\nu(s;v,w) = \pi^{-v}\,\frac{\Gamma\big(\frac{v+1-s}{2}\big)\Gamma\big(\frac{v+w-s}{2}\big)\Gamma\big(\frac{v+s}{2}\big)\Gamma\big(\frac{v+w+s-1}{2}\big)}{\Gamma\big(\frac{w}{2}\big)\Gamma\big(v+\frac{w}{2}\big)} \tag{4.2}$$

and at complex places ν

$$\mathcal{G}_\nu(s;v,w) = (2\pi)^{-2v}\,\frac{\Gamma(v+1-s)\Gamma(v+w-s)\Gamma(v+s)\Gamma(v+w+s-1)}{\Gamma(w)\Gamma(2v+w)} \tag{4.3}$$

In these expressions $i\mu_{f,\nu}$ and $-i\mu_{f,\nu}$ are the local parameters of the spherical principal series representation generated by f at ν. That these integrals are ratios of products of gamma functions is an elementary computation: see [DG1] for the real case and [DG2] for the complex case, invoking uniqueness of local Whittaker models.

In these spherical cases, the Whittaker functions are readily expressible in terms of the classical K-Bessel functions, as

$$W_\nu \begin{pmatrix} a & \\ & 1 \end{pmatrix} = \begin{cases} |a|^{1/2} K_{i\mu_{f,\nu}}(2\pi|a|) & (\text{for } \nu \approx \mathbb{R}) \\ |a|\, K_{2i\mu_{f,\nu}}(4\pi|a|) & (\text{for } \nu \approx \mathbb{C}) \end{cases}$$

Having computed the integrals $\langle \text{Pé}, F\rangle$ of the Poincaré series against cuspforms F, with respect to an orthonormal basis $\{F\}$ of everywhere locally spherical cuspforms, the cuspidal part of the spectral decomposition of the Poincaré series should be

$$\sum_F \langle \text{Pé}, F\rangle \cdot F \;=\; \sum_F \;\bar{\rho}_F\, \mathcal{G}_{F_\infty}(v,w)\, L(v+\tfrac{1}{2}, \overline{F}) \cdot F$$

where the archimedean factors are grouped together as

$$\mathcal{G}_{F_\infty}(v,w) \;=\; \prod_{\nu|\infty} \mathcal{G}_\nu(\tfrac{1}{2} + i\bar{\mu}_{F,\nu}; v, w)$$

with $\mathcal{G}_\nu$ as in (4.2) and (4.3) for each F, with all ambiguous constants at infinite places absorbed into $\bar{\rho}_F$. Traditionally, the constant ρ_F is denoted by $\rho_F(1)$, referring to its appearance as the first classical (numerical) Fourier coefficient of F. As mentioned at the beginning of this section, and as demonstrated shortly, a modified form Pé* of the Poincaré series Pé(g), obtained by subtracting a suitable Eisenstein series, *is* L^2. From this it will follow that the above spectral sum is the discrete L^2 part of Pé.

Since Pé* is square-integrable for $\Re(w)$ and $\Re(v)$ large, the sum of projections to cuspforms is certainly convergent in L^2 for such v, w. In fact, for arbitrary v, w, the sum of projections to cuspforms converges. In essence, the convergence follows from the fact that $\overline{\rho}_F \cdot \mathcal{G}_{F_\infty}(v,w)$ has exponential decay in the archimedean parameters of F. To see this, consider the usual integral representation, against an Eisenstein series, of the completed $GL_2 \times GL_2$ Rankin-Selberg L–function $\Lambda(s, F \otimes \bar{F})$ from [J]. For the general case of $GL_m \times GL_n$, see the literature review in [CPS2]. Taking the residue at $s = 1$ gives

$$\text{residue of Eisenstein series} \;=\; |\rho_F|^2 \cdot L_\infty(1, F\otimes\bar{F}) \cdot \operatorname*{Res}_{s=1} L(s, F\otimes\bar{F}) \qquad (\text{with } |F|_{L^2} = 1)$$

The constant on the left is manifestly independent of F. The local factors of the finite-prime L–function $L(s, F\otimes\bar{F})$ on the right obtained from the integral representation differ from those of the *correct* convolution L–function obtained from the local theory at only absolutely ramified primes in k, and the discrepancies are readily estimated. This gives

$$|\rho_F|^2 \;=\; \frac{\text{residue of Eisenstein series}}{L_\infty(1, F\otimes\bar{F}) \cdot \operatorname*{Res}_{s=1} L(s, F\otimes\bar{F})}$$

Comparing $L_\infty(1, F\otimes\bar{F})$ with $\mathcal{G}_{F_\infty}(v,w)$ using Stirling's formula, the ratio

$$\frac{|\mathcal{G}_{F_\infty}(v,w)|}{|L_\infty(1, F\otimes\bar{F})|^{1/2}}$$

has *exponential decay* in the archimedean parameters of F. Although it is far more than we need, a *sharp* lower bound for the residue at $s = 1$ can be obtained by combining [HR] and [Ba]. Finally, a routine convexity bound implies that $L(\frac{1}{2} + v, \overline{F})$ grows at worst polynomially in the archimedean data of F. The number of cuspforms with archimedean data within a given bound grows polynomially, from Weyl's Law [LV], or from the upper bound of [Do]. Thus, the spectral sum is absolutely convergent for $(v, w) \in \mathbb{C}^2$, except for the poles of $\mathcal{G}_{F_\infty}(v, w)$.

For the remaining decomposition, subtract (as in [DG1], [DG2], in a classical setting) an Eisenstein series from the Poincaré series, leaving a function in L^2 with sufficient decay to be integrated against Eisenstein series. The correct Eisenstein series to subtract is visible from the dominant part of the constant term of the Poincaré series, as follows. Write the Poincaré series as

$$\text{Pé}(g) \;=\; \sum_{\gamma \in M_k \backslash G_k} \varphi(\gamma g) \;=\; \sum_{\gamma \in P_k \backslash G_k} \sum_{\beta \in N_k} \varphi(\beta \gamma g)$$

By Poisson summation

$$\text{Pé}(g) \;=\; \sum_{\gamma \in P_k \backslash G_k} \sum_{\psi \in (N_k \backslash N_{\mathbb{A}})^\frown} \widehat{\varphi}_{\gamma g}(\psi) \tag{4.4}$$

where $\varphi_g(n) = \varphi(ng)$, and $\widehat{\varphi}$ is the Fourier transform along $N_{\mathbb{A}}$. The trivial–ψ (that is, with $\psi = 1$) Fourier term

$$\sum_{\gamma \in P_k \backslash G_k} \widehat{\varphi}_{\gamma g}(1) \tag{4.5}$$

is an Eisenstein series, since the function

$$g \longrightarrow \widehat{\varphi}_g(1) = \int_{N_{\mathbb{A}}} \varphi(ng)\, dn$$

is left $M_{\mathbb{A}}$–equivariant by the character $\delta\chi_0$, and left $N_{\mathbb{A}}$–invariant.

For $\xi \in M_k$,

$$\begin{aligned}
\widehat{\varphi}_{\xi g}(\psi) \;&=\; \int_{N_{\mathbb{A}}} \overline{\psi}(n)\, \varphi(n \xi g)\, dn \\
&=\; \int_{N_{\mathbb{A}}} \overline{\psi}(n)\, \varphi(\xi \cdot \xi^{-1} n \xi \cdot g)\, dn \;=\; \int_{N_{\mathbb{A}}} \overline{\psi}(\xi n \xi^{-1})\, \varphi(n \cdot g)\, dn \;=\; \widehat{\varphi}_g(\psi^\xi)
\end{aligned}$$

where $\psi^\xi(n) = \psi(\xi n \xi^{-1})$, by replacing n by $\xi n \xi^{-1}$, using the left M_k–invariance of φ, and invoking the product formula to see that the change-of-measure is trivial. Since this action of $Z_k \backslash M_k$ is transitive on non-trivial characters on $N_k \backslash N_{\mathbb{A}}$, for a fixed choice of non-trivial character ψ, the sum over non-trivial characters can be rewritten as a more familiar sort of Poincaré series

$$\begin{aligned}
\sum_{\gamma \in P_k \backslash G_k} \sum_{\psi' \in (N_k \backslash N_{\mathbb{A}})^\frown} \widehat{\varphi}_{\gamma g}(\psi') \;&=\; \sum_{\gamma \in P_k \backslash G_k} \sum_{\xi \in Z_k \backslash M_k} \widehat{\varphi}_{\gamma g}(\psi^\xi) \\
&=\; \sum_{\gamma \in P_k \backslash G_k} \sum_{\xi \in Z_k \backslash M_k} \widehat{\varphi}_{\xi \gamma g}(\psi) \;=\; \sum_{\gamma \in Z_k N_k \backslash G_k} \widehat{\varphi}_{\gamma g}(\psi)
\end{aligned}$$

Denote this version of the Poincaré series, with the Eisenstein series subtracted, by

$$\text{Pé}^*(g) \;=\; \sum_{\gamma\in Z_k N_k\backslash G_k} \widehat{\varphi}_{\gamma g}(\psi) \;=\; \text{Pé}(g) \;-\; \sum_{\gamma\in P_k\backslash G_k} \widehat{\varphi}_{\gamma g}(1) \tag{4.6}$$

Remark: With (4.6), the square integrability of the Poincaré series in Theorem 2.7 is that, for φ admissible, the *modified* Poincaré series $\text{Pé}^*(g)$ is in $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}})$.

Now we describe the continuous part of the spectral decomposition. At every place ν, let η_ν be the spherical vector in the (non-normalized) principal series $\text{Ind}_{P_\nu}^{G_\nu}\chi_\nu$, with $\eta_\nu(1) = 1$. Take $\eta = \bigotimes_{\nu\le\infty}\eta_\nu$. The corresponding Eisenstein series is

$$E_\chi(g) \;=\; \sum_{\gamma\in P_k\backslash G_k} \eta(\gamma g)$$

For any left $Z_{\mathbb{A}}G_k$–invariant and right $K_{\mathbb{A}}$–invariant square-integrable F on $G_{\mathbb{A}}$, write

$$\langle F, E_\chi\rangle = \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} F(g)\,\overline{E_\chi(g)}\,dg$$

With suitable normalization of measures,

$$\text{continuous-spectrum part of } F \;=\; \int_{\Re(\chi)=\frac{1}{2}} \langle F, E_\chi\rangle\; E_\chi\, d\chi$$

Explicitly, let

$$\kappa = \text{meas}(\mathbb{J}^1/k^\times) \tag{4.7}$$

where the measure on $\mathbb{J}^1/k^\times$ is the image of the measure γ on $\mathbb{J}$ defined in [W2], page 128. From [W2], page 129, Corollary, the residue of the zeta-function of k at $s = 1$ is

$$\operatorname*{Res}_{s=1} \zeta_k(s) \;=\; \frac{\kappa}{|D_k|^{\frac{1}{2}}}$$

where D_k is the discriminant of k. Then,

$$\text{continuous-spectrum part of } F \;=\; \frac{1}{4\pi i\kappa}\sum_{\chi} \int_{\Re(s)=\frac{1}{2}} \langle F, E_{s,\chi}\rangle\cdot E_{s,\chi}\; ds$$

where the sum is over all *absolutely* unramified characters $\chi \in \widehat{C}_0$. Here $E_{s,\chi} = E_{\chi\,|\cdot|^s}$. In general, this requires the isometric extension to L^2 of integral formulas that do not converge on all of L^2, but do converge on the dense subspace of pseudo-Eisenstein series with compactly supported data, as in [Go2], for example.

In our situation, $\text{Pé}^*(g)$ is smooth and it and its derivatives are of sufficient decay for $\Re(w)$ and $\Re(v)$ large, so the integrals against Eisenstein series, with parameter in a bounded vertical strip containing the critical line, converge absolutely. For the same reasons, the continuous part of its spectral decomposition converges: this will be explicit in the computations below.

There is no residual spectrum component since residual automorphic forms on GL_2 are associated to one-dimensional representations, which have no Whittaker models. Thus, by Theorem 2.7 (see also the remark above), by Proposition 4.1 and (4.6), with respect to an orthonormal basis $\{F\}$ of everywhere locally spherical cuspforms, there is the spectral decomposition (with no residual component)

$$\text{Pé} \;=\; \left(\int_{N_\infty} \varphi_\infty\right) \cdot E_{v+1} \;+\; \sum_F \overline{\rho}_F \cdot \mathcal{G}_{F_\infty}(v,w) \cdot L(v+\tfrac{1}{2}, \overline{F}) \cdot F$$
$$+\; \frac{1}{4\pi i\kappa} \sum_\chi \int_{\Re(s)=\frac{1}{2}} \langle \text{Pé}^*, E_{s,\chi}\rangle \cdot E_{s,\chi}\, ds \tag{4.8}$$

where E_s is $E_{s,1}$. To compute the pairing $\langle \text{Pé}^*, E_{s,\chi}\rangle$ in the continuous part, first consider an Eisenstein series

$$E(g) \;=\; \sum_{\gamma\in P_k\backslash G_k} \eta(\gamma g)$$

for η left P_k–invariant, left $M_{\mathbb{A}}$–equivariant and left $N_{\mathbb{A}}$–invariant. The Fourier expansion of this Eisenstein series is

$$E(g) \;=\; \sum_{\psi\in(N_k\backslash N_{\mathbb{A}})^\frown} \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}(n)\, E(ng)\, dn$$

For a fixed non-trivial character ψ, the ψ^{th} Fourier term is

$$\int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}(n)\, E(ng)\, dn \;=\; \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}(n) \sum_{\gamma\in P_k\backslash G_k} \eta(\gamma n g)\, dn$$

$$=\; \sum_{w\in P_k\backslash G_k/N_k} \int_{(N_k\,\cap\, w^{-1}P_k w)\backslash N_{\mathbb{A}}} \overline{\psi}(n)\, \eta(wng)\, dn$$

$$=\; \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}(n)\, \eta(ng)\, dn \;+\; \int_{N_{\mathbb{A}}} \overline{\psi}(n)\, \eta(w_\circ ng)\, dn$$

$$=\; 0 \;+\; \int_{N_{\mathbb{A}}} \overline{\psi}(n)\, \eta(w_\circ ng)\, dn \qquad \left(\text{where } w_\circ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

because ψ is non-trivial and η is left $N_{\mathbb{A}}$–invariant. Denote the ψ^{th} Fourier term by

$$W^E(g) \;=\; W^E_{\eta,\,\psi}(g) \;=\; \int_{N_{\mathbb{A}}} \overline{\psi}(n)\, \eta(w_\circ ng)\, dn \tag{4.9}$$

Proposition 4.10. *Fix $s \in \mathbb{C}$ with $\Re(s) > 1$, and let $\varphi_\infty \in \Omega_\ell$ with $\Re(v)$, ℓ sufficiently large. Then*

$$\langle \text{Pé}^*, E_{s,\chi}\rangle \;=\; \overline{\chi}(\mathfrak{d}) \left(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \cdot \overline{W}^E_{s,\,\chi,\,\infty} \right) \frac{L(v+\bar{s},\overline{\chi}) \cdot L(v+1-\bar{s},\chi)}{L(2\bar{s},\overline{\chi}^2)} \cdot |\mathfrak{d}|^{-(v-\bar{s}+1/2)}$$

where $\mathfrak{d}$ is a differental idele ([W2]*, page 113, Definition 4) with component 1 at archimedean places.*

Proof: Fix non-trivial ψ on $N_k\backslash N_{\mathbb{A}}$. For $\Re(v)$ and ℓ both large, the modified Poincaré series $\text{Pé}^*(g)$ has sufficient (polynomial) decay, so that we can unwind it to obtain (see (4.6))

$$\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \text{Pé}^*(g)\, \overline{E}_{s,\chi}(g)\, dg \;=\; \int_{Z_{\mathbb{A}}N_{\mathbb{A}}\backslash G_{\mathbb{A}}} \int_{N_k\backslash N_{\mathbb{A}}} \widehat{\varphi}_{ng}(\psi)\, \overline{E}_{s,\chi}(ng)\, dn\, dg \tag{4.11}$$

$$= \int_{Z_{\mathbb{A}}N_{\mathbb{A}}\backslash G_{\mathbb{A}}} \widehat{\varphi}_g(\psi) \int_{N_k\backslash N_{\mathbb{A}}} \psi(n)\overline{E}_{s,\chi}(ng)\, dn\, dg \;=\; \int_{Z_{\mathbb{A}}N_{\mathbb{A}}\backslash G_{\mathbb{A}}} \widehat{\varphi}_g(\psi)\, \overline{W}^E_{s,\chi}(g)\, dg$$

Since

$$\widehat{\varphi}_g(\psi) \;=\; \int_{N_{\mathbb{A}}} \overline{\psi}(n)\, \varphi(ng)\, dn$$

the last integral in (4.11) is

$$\begin{aligned} \int_{Z_{\mathbb{A}}N_{\mathbb{A}}\backslash G_{\mathbb{A}}} \int_{N_{\mathbb{A}}} \overline{\psi}(n)\, \varphi(ng)\, \overline{W}^E_{s,\chi}(g)\, dn\, dg \;&=\; \int_{Z_{\mathbb{A}}N_{\mathbb{A}}\backslash G_{\mathbb{A}}} \int_{N_{\mathbb{A}}} \varphi(ng)\, \overline{W}^E_{s,\chi}(ng)\, dn\, dg \\ &= \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\, \overline{W}^E_{s,\chi}(g)\, dg \end{aligned} \tag{4.12}$$

The Whittaker function of the Eisenstein series factors over primes, into local factors depending only upon the local data at ν,

$$W^E_{s,\chi} \;=\; \bigotimes_\nu W^E_{s,\chi,\nu}$$

Thus, by (4.11) and (4.12),

$$\langle \text{Pé}^*, E_{s,\chi}\rangle \;=\; \left(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \cdot \overline{W}^E_{s,\chi,\infty} \right) \cdot \prod_{\nu<\infty} \int_{Z_\nu\backslash G_\nu} \varphi_\nu(g_\nu)\, \overline{W}^E_{s,\chi,\nu}(g_\nu)\, dg_\nu$$

At finite ν, using an Iwasawa decomposition and the vanishing of φ_ν off $M_\nu K_\nu$ (see (3.2)), as in the integration against cuspforms, the local factor is

$$\int_{k_\nu^\times} |a|_\nu^v\, \overline{W}^E_{s,\chi,\nu} \begin{pmatrix} a & \\ & 1 \end{pmatrix} da$$

However, for Eisenstein series, the natural normalization of the Whittaker functions differs from that used for cuspforms, instead presenting the local Whittaker functions as images under intertwining operators. Specifically, define the *normalized spherical vector* for data s, χ_ν to be

$$\eta_\nu(pk) = |a/d|_\nu^s \cdot \chi_\nu(a/d) \qquad \left(\text{for } p = \begin{pmatrix} a & * \\ & d \end{pmatrix} \in P_\nu \text{ and } k \in K_\nu \right)$$

The corresponding spherical local Whittaker function for Eisenstein series is the meromorphically continued integral (see (4.9))

$$W^E_{s,\chi,\nu}(g) = \int_{N_\nu} \overline{\psi}_\nu(n)\, \eta_\nu(w_\circ ng)\, dn$$

The Mellin transform of the Eisenstein-series normalization $W^E_{s,\chi,\nu}$ is readily compared to the Mellin transform of the usual normalization as follows. Let $\mathfrak{d}_\nu \in k_\nu^\times$ be such that

$$(\mathfrak{o}^*_\nu)^{-1} = \mathfrak{d}_\nu \cdot \mathfrak{o}_\nu$$

Let $\mathfrak{d}$ be the idele with ν^{th} component $\mathfrak{d}_\nu$ at finite places ν and component 1 at archimedean places. Then for finite ν the ν^{th} local integral is (see Appendix 2 for details),

$$\int_{k_\nu^\times} |a|^v_\nu\, \overline{W}^E_{s,\chi,\nu}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da \;=\; |\mathfrak{d}_\nu|^{1/2}_\nu \cdot \frac{L_\nu(v+\bar{s},\overline{\chi}_\nu)\cdot L_\nu(v+1-\bar{s},\chi_\nu)}{L_\nu(2\bar{s},\overline{\chi}^2_\nu)} \cdot |\mathfrak{d}_\nu|_\nu^{-(v+1-\bar{s})}\, \overline{\chi}_\nu(\mathfrak{d}_\nu)$$

and the proposition follows. $\square$

Accordingly, the spectral decomposition (4.8) is

(4.13)

$$\text{Pé} \;=\; \left(\int_{N_\infty} \varphi_\infty\right) \cdot E_{v+1} \;+\; \sum_F \left(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \cdot \overline{W}_{F,\infty}\right) \cdot L(v+\tfrac{1}{2},\overline{F}) \cdot F$$

$$+ \sum_\chi \frac{\overline{\chi}(\mathfrak{d})}{4\pi i\kappa} \int_{\Re(s)=\frac{1}{2}} \left(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \cdot W^E_{1-s,\overline{\chi},\infty}\right) \frac{L(v+1-s,\overline{\chi})\cdot L(v+s,\chi)}{L(2-2s,\overline{\chi}^2)}\, |\mathfrak{d}|^{-(v+s-1/2)} \cdot E_{s,\chi}\, ds$$

where we replaced $\bar{s}$ by $1-s$, for $\Re(s) = \frac{1}{2}$, to maintain holomorphy of the integrand. The archimedean-place Whittaker functions can be expressed in terms of the usual K–Bessel function as follows. Let

$$\eta_\nu(nmk) = \eta_{s,\nu}(nmk) = |a/d|^s_\nu \qquad \left(\text{for } n\in N_\nu,\ m = \begin{pmatrix} a & \\ & d \end{pmatrix} \in M_\nu,\ k \in K_\nu\right)$$

The normalization of the Whittaker function is

$$W^E_{s,\nu}(g) = \int_{N_\nu} \overline{\psi}_\nu(n)\, \eta_\nu(w_\circ ng)\, dn \qquad (\text{for } \Re(s) \gg 0 \text{ and fixed non-trivial } \psi)$$

Then, for ν archimedean and fixed non-trivial character $\psi_{0,\nu}$ on k_ν

$$W^E_{s,\nu}\begin{pmatrix} a & \\ & 1 \end{pmatrix} = \int_{k_\nu} \overline{\psi}_{0,\nu}(x) \left|\frac{a}{aa^\iota + xx^\iota}\right|^s_\nu dx = |a|^{1-s}_\nu \int_{k_\nu} \overline{\psi}_{0,\nu}(ax)\, \frac{1}{|1+xx^\iota|^s_\nu}\, dx$$

by replacing x by ax, where ι is the complex conjugation for $\nu \approx \mathbb{C}$ and the identity map for $\nu \approx \mathbb{R}$. The usual computation shows that

$$W^E_{s,\mathbb{R}}\begin{pmatrix} a & \\ & 1 \end{pmatrix} = \frac{|a|^{1/2}}{\pi^{-s}\Gamma(s)} \int_0^\infty e^{-\pi(t+\frac{1}{t})|a|}\, t^{s-\frac{1}{2}}\, \frac{dt}{t} = \frac{2\,|a|^{1/2} K_{s-1/2}(2\pi|a|)}{\pi^{-s}\Gamma(s)}$$

and, similarly (with the classical measure doubled, as in Iwasawa-Tate theory)

$$W^E_{s,\mathbb{C}}\begin{pmatrix} a & \\ & 1\end{pmatrix} = \frac{|a|}{(2\pi)^{-2s}\Gamma(2s)}\int_0^\infty e^{-2\pi(t+\frac{1}{t})|a|}\, t^{2s-1}\,\frac{dt}{t} = \frac{2\,|a|K_{2s-1}(4\pi|a|)}{(2\pi)^{-2s}\Gamma(2s)}$$

To simplify the integral over $Z_\infty\backslash G_\infty$ in the continuous part of (4.13), let

$$\Phi_\nu(x) = \varphi_\nu\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} \qquad \text{(for } \nu \text{ archimedean)}$$

Using the right K_ν–invariance and an Iwasawa decomposition,

$$\begin{aligned}\text{(4.14)} \qquad \int_{Z_\nu\backslash G_\nu} \varphi_\nu\cdot W^E_{s,\nu} &= \int_{k_\nu^\times}\int_{k_\nu} |a|_\nu^v\,\Phi_\nu(x)\,W^E_{s,\nu}\begin{pmatrix} a & \\ & 1\end{pmatrix}\psi_{0,\nu}(ax)\,dx\,da \\ &= \int_{k_\nu^\times} |a|_\nu^v\,\widehat{\Phi}_\nu(a)\,W^E_{s,\nu}\begin{pmatrix} a & \\ & 1\end{pmatrix} da\end{aligned}$$

For $\chi \in \widehat{C}_0$ absolutely unramified, we have $W^E_{s,\chi,\nu} = W^E_{s+it_\nu,\nu}$, where $t_\nu \in \mathbb{R}$ is the parameter of the local component χ_ν of χ. Then all archimedean integrals in the continuous part of the spectral decomposition of Pé* are given by (4.14) with s replaced by $1-s-it_\nu$.

In particular, for φ_ν specialized to (3.11),

$$\text{(4.15)} \qquad \int_{Z_\nu\backslash G_\nu} \varphi_\nu\cdot W^E_{s,\nu} = \begin{cases} \dfrac{\mathcal{G}_\nu(s;v,w)}{\pi^{-s}\Gamma(s)} & \text{if } \nu\approx\mathbb{R} \\ \dfrac{\mathcal{G}_\nu(s;v,w)}{(2\pi)^{-2s-1}\Gamma(2s)} & \text{for } \nu\approx\mathbb{C}\end{cases}$$

where $\mathcal{G}_\nu(s;v,w)$ is given in (4.2) and (4.3). Furthermore, with these choices of φ_ν,

$$\text{(4.16)} \qquad \int_{N_\nu}\varphi_\nu = \begin{cases} \sqrt{\pi}\,\dfrac{\Gamma(\frac{w-1}{2})}{\Gamma(\frac{w}{2})} & \text{for } \nu\approx\mathbb{R} \\ 2\pi\,(w-1)^{-1} & \text{if } \nu\approx\mathbb{C}\end{cases}$$

As usual, let r_1 and r_2 denote the number of real and complex embeddings of k, respectively. Following [DG1], Proposition 5.10, we now prove

Theorem 4.17. *Let φ be as in (3.11). Then the Poincaré series Pé(g) has meromorphic continuation to a region in $\mathbb{C}^2$ containing $v=0$, $w=1$. As a function of w, for $v=0$, it is holomorphic in the half-plane $\Re(w) > 11/18$, except for $w=1$ where it has a pole of order r_1+r_2+1.*

Proof: Let Pé$^*_{\rm cusp}$ and Pé$^*_{\rm cont}$ be, respectively, the discrete and continuous parts of Pé*. Then the spectral decomposition (4.13) is

$$\text{Pé} \;=\; R(w)\cdot E_{v+1} \;+\; \text{Pé}^*_{\rm cusp} \;+\; \text{Pé}^*_{\rm cont} \qquad \left(\text{where } R(w) = \int_{N_\infty}\varphi_\infty\right)$$

the integral being computed by (4.16). As in the proof of Proposition 4.1, the series giving $\mathrm{Pé}^*_{\mathrm{cusp}}$

$$\sum_F \bar{\rho}_F\, \mathcal{G}_{F_\infty}(v,w)\, L(v+\tfrac{1}{2},\overline{F})\cdot F$$

converges absolutely for $(v,w)\in\mathbb{C}^2$, away from the poles of $\mathcal{G}_\nu(\frac{1}{2}+i\bar{\mu}_{F,\nu};v,w)$. The fact that $\mathrm{Pé}^*_{\mathrm{cusp}}$ is equal to this spectral sum follows from the square integrability of $\mathrm{Pé}^*$ for $\Re(w)>1$ and large $\Re(v)$ (see Theorem 2.7, (4.6) and Appendix 1). Furthermore, using (4.2), (4.3) and the Kim-Shahidi bound for the local parameters $|\Re(i\mu_{f,\nu})|<1/9$ (see [K], [KS]), the cuspidal part $\mathrm{Pé}^*_{\mathrm{cusp}}$ is holomorphic for $v=0$ and $\Re(w)>11/18$.

Estimates for the continuous part are easier than those for the cuspidal part: the most delicate feature, the Siegel-zero-type estimates from [HR], are replaced by the easier de la Vallée-Poussin or Hadamard-type lower bounds for GL_1 L–functions on $\Re(s)=1$, and by trivial convexity bounds for the L–functions in the numerator. Thus, the integrands in the integrals in (4.13) have enough decay in the parameters to ensure absolute convergence of the integral and sum over χ. Also, note that $\mathrm{Pé}^*_{\mathrm{cont}}$ is holomorphic for $\Re(v)>\frac{1}{2}$ and $\Re(w)>1$. Aiming to analytically continue to $v=0$, first take $\Re(v)=1/2+\varepsilon$, and move the line of integration from $\sigma=1/2$ to $\sigma=1/2-2\varepsilon$. This picks up the residue of the integrand corresponding to χ trivial, due to the pole of $\zeta_k(v+s)$ at $v+s=1$, that is, at $s=1-v$. Its contribution is

$$\frac{1}{2}\,Q(v;v,w)\;\cdot\;|\mathfrak{d}|^{1/2}\cdot|\mathfrak{d}|^{-1/2}\cdot E_{1-v}\;=\;\frac{1}{2}\,Q(v;v,w)\cdot E_{1-v}$$

where

$$Q(s;v,w)\;=\;\int_{Z_\infty\backslash G_\infty}\varphi_\infty\cdot W^E_{s,\infty}$$

is the ratio of products of gamma functions computed by (4.15). This expression of $\mathrm{Pé}^*_{\mathrm{cont}}$ is holomorphic in v in the strip

$$\frac{1}{2}-\varepsilon\le\Re(v)\le\frac{1}{2}+\varepsilon$$

Now, take v with $\Re(v)=1/2-\varepsilon$, and then move the vertical integral from $\sigma=1/2-2\varepsilon$ back to $\sigma=1/2$. This picks up (-1) times the residue at the pole of $\zeta_k(v+1-s)$ at 1, that is, at $s=v$, with another sign due to the sign of s inside this zeta function. Thus, we pick up the residue

$$\frac{1}{2}\,Q(1-v;v,w)\cdot\frac{\zeta_k(2v)}{\zeta_k(2-2v)}\cdot|\mathfrak{d}|^{-2v+1}\cdot E_v\;=\;\frac{1}{2}\,Q(1-v;v,w)\cdot\frac{\zeta_\infty(2-2v)}{\zeta_\infty(2v)}\cdot E_{1-v}$$

where the last identity was obtained from the functional equation of the Eisenstein series E_v. Since $\mathcal{G}_\nu(s;v,w)$ defined in (4.2) and (4.3) is invariant under $s\longrightarrow 1-s$, it follows by (4.15) that the above residues are equal. Note that the part of $\mathrm{Pé}^*_{\mathrm{cont}}$ corresponding to the vertical line integral and the sum over χ is now holomorphic in a region of $\mathbb{C}^2$ containing $v=0$, $w=1$. In particular, for $v=0$, this part of the continuous spectrum is holomorphic in the half-plane $\Re(w)>1/2$.

On the other hand, by direct computation, the *apparent* pole of $R(w)E_{v+1}$ at $v=0$ (independent of w) cancels the corresponding pole of $Q(v;v,w)E_{1-v}$. To establish that the order of the pole at $w=1$, when $v=0$, is r_1+r_2+1, consider the most relevant terms (recall (4.15), (4.16)) in the Laurent expansions of $R(w)E_{v+1}$ and $Q(v;v,w)E_{1-v}$. Putting them together, we obtain an expression

$$\frac{1}{v}\cdot\left[\frac{c_1}{(w-1)^{r_1+r_2}}\;-\;\frac{c_2}{(2v+w-1)^{r_1+r_2}}\right]$$

for some constants c_1, c_2. As there is no pole at $v=0$, we have $c_1=c_2$. Canceling the factor $1/v$, and then setting $v=0$, the assertion follows.

This completes the proof. □

§5. Asymptotics

Let k be a number field with r_1 real places and r_2 complex places. Let φ be as in (3.11). By Theorem 3.12, for $\Re(v)$ and $\Re(w)$ sufficiently large, the integral $I(\chi_0)=I(v,w)$ defined by (3.6) is

$$I(v,w) \;=\; \sum_{\chi\in\widehat{C}_{0,S}} \frac{1}{2\pi i}\int\limits_{\Re(s)=\sigma} L(1-s+v,,\, f_1\otimes\overline{\chi})\; L(s,\,\bar{f}_2\otimes\chi)\,\mathcal{K}_\infty(s,\,v,\,w,\,\chi)\,ds \tag{5.1}$$

where $\mathcal{K}_\infty(s,\,v,\,w,\,\chi)$ is given by (3.9) and (3.10), and where the sum is over $\chi\in\widehat{C}_0$ unramified outside S and with bounded ramification, depending only upon f_1 and f_2.

By Theorem 4.17, $I(v,w)$ has meromorphic continuation to a region in $\mathbb{C}^2$ containing $v=0$, $w=1$. In particular, for $f_1=f_2=\bar{f}$, then $I(0,w)$ is holomorphic for $\Re(w)>11/18$, except for $w=1$ where it has a pole of order r_1+r_2+1.

We want to shift the line of integration to $\Re(s)=\frac{1}{2}$ in (5.1) and set $v=0$. To do so, we need an analytic continuation and reasonable decay in $|\Im(s)|$ for the kernel function $\mathcal{K}_\infty(s,\,v,\,w,\,\chi)$. In fact, for applications, we want *precise* asymptotics as the parameters s, v, w, χ vary. By the decomposition (3.10), the analysis of the kernel $\mathcal{K}_\infty(s,\,v,\,w,\,\chi)$ reduces to the corresponding analysis of the local component $\mathcal{K}_\nu(s,\,v,\,w,\,\chi_\nu)$, for $\nu|\infty$. For ν complex, the asymptotic formula in [DG2] Theorem 6.2 suffices. For coherence, the simple computation is included which matches, as it should, the local integral (3.9), for ν complex, the integral (4.15) in [DG2].

Fix a complex place ν. Every irreducible unitary representation of $GL_2(\mathbb{C})$ is a principal series representation (see [GJ], [GGPS]), and the spherical ones are spherical principal series. Recall that any character χ_ν of $Z_\nu\backslash M_\nu\approx\mathbb{C}^\times$ has the form

$$\chi_\nu(m_\nu) \;=\; |z_\nu|_{\mathbb{C}}^{\frac{\ell_\nu}{2}+it_\nu}\, z_\nu^{-\ell_\nu} \qquad \left(m_\nu=\begin{pmatrix} z_\nu & 0\\ 0 & 1\end{pmatrix},\; t_\nu\in\mathbb{R},\; \ell_\nu\in\mathbb{Z}\right)$$

Then, the local integral (3.9) at ν is of the form

$$\begin{aligned}\mathcal{K}_\nu(s,\,v,\,w,\,\chi_\nu) \;=\; &\int\limits_0^\infty\int\limits_0^\infty\int\limits_{\mathbb{C}}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} \left(|x|^2+1\right)^{-w}\, e^{2\pi i\cdot \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(a_1xe^{i\theta_1}-a_2xe^{i\theta_2})}\\ &\cdot a_1^{2v+1-2s-2it_\nu}K_{2i\mu_1}(4\pi a_1)\,a_2^{2s+2it_\nu-1}K_{2i\bar{\mu}_2}(4\pi a_2)\,e^{i\ell_\nu\theta_1}e^{-i\ell_\nu\theta_2}\;d\theta_1 d\theta_2\,dx\,da_1da_2\end{aligned}$$

Replacing x by x/a_1, we obtain

$$\begin{aligned}\mathcal{K}_\nu(s,\,v,\,w,\,\chi_\nu) \;=\; &\int\limits_0^\infty\int\limits_0^\infty\int\limits_{\mathbb{C}}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} \left(\frac{a_1}{\sqrt{|x|^2+a_1^2}}\right)^{2w} e^{2\pi i\cdot \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\left(xe^{i\theta_1}-\frac{a_2}{a_1}xe^{i\theta_2}\right)}\\ &\cdot a_1^{2v-1-2s-2it_\nu}K_{2i\mu_1}(4\pi a_1)\,a_2^{2s+2it_\nu-1}K_{2i\bar{\mu}_2}(4\pi a_2)\,e^{i\ell_\nu\theta_1}e^{-i\ell_\nu\theta_2}\;d\theta_1 d\theta_2\,dx\,da_1da_2\end{aligned}$$

Upon further substituting

$$a_1=r\cos\phi \quad x_1=r\sin\phi\cos\theta \quad x_2=r\sin\phi\sin\theta \quad a_2=u\cos\phi$$

with $0 \le \phi \le \frac{\pi}{2}$ and $0 \le \theta \le 2\pi$, then

$$\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) \;=\; \int_0^\infty \int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (\cos\phi)^{2w+2v-1}\, e^{2\pi i\cdot \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(r\sin\phi\cdot e^{i(\theta+\theta_1)} - u\sin\phi\cdot e^{i(\theta+\theta_2)})}$$
$$\cdot\, r^{2v+1-2s-2it_\nu} K_{2i\mu_1}(4\pi r\cos\phi)\, u^{2s+2it_\nu-1} K_{2i\bar{\mu}_2}(4\pi u\cos\phi)\, e^{i\ell_\nu\theta_1} e^{-i\ell_\nu\theta_2} \sin\phi\; d\theta_1 d\theta_2\, d\theta\, d\phi\, dr\, du$$

Using the Fourier expansion

$$e^{it\sin\theta} \;=\; \sum_{k=-\infty}^{\infty} J_k(t)\, e^{ik\theta}$$

we obtain

$$\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) = (2\pi)^3 \int_0^\infty \int_0^\infty \int_0^{\frac{\pi}{2}} K_{2i\mu_1}(4\pi r\cos\phi) K_{2i\bar{\mu}_2}(4\pi u\cos\phi) J_{\ell_\nu}(4\pi r\sin\phi) J_{\ell_\nu}(4\pi u\sin\phi)$$
$$\cdot\, u^{2s+2it_\nu} r^{2v+2-2s-2it_\nu} (\cos\phi)^{2w+2v-1} \sin\phi\; \frac{d\phi dr du}{ru}$$

In the notation of [DG2], equation (4.15), this is essentially $\mathcal{K}_{\ell_\nu}(2s+2it_\nu, 2v, 2w)$. It follows that $\mathcal{K}_\nu(s,v,w,\chi_\nu)$ is analytic in a region $\mathcal{D}: \Re(s) = \sigma > \frac{1}{2} - \varepsilon_0$, $\Re(v) > -\varepsilon_0$ and $\Re(w) > \frac{3}{4}$, with a fixed (small) $\varepsilon_0 > 0$, and moreover, we have the asymptotic formula

$$\begin{aligned} \mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) &= \pi^{-2v+1} A(v,w,\mu_1,\mu_2) \cdot \left(1 + \ell_\nu^2 + 4(t+t_\nu)^2\right)^{-w} \\ &\quad \cdot \left[1 \;+\; \mathcal{O}_{\sigma,\, v,\, w,\, \mu_1,\, \mu_2}\left(\left(\sqrt{1+\ell_\nu^2+4(t+t_\nu)^2}\right)^{-1}\right)\right] \end{aligned} \tag{5.2}$$

where $A(v,w,\mu_1,\mu_2)$ is the ratio of products of gamma functions

$$2^{2w-2v-4}\, \frac{\Gamma(w+v+i\mu_1+i\bar{\mu}_2)\Gamma(w+v-i\mu_1+i\bar{\mu}_2)\Gamma(w+v+i\mu_1-i\bar{\mu}_2)\Gamma(w+v-i\mu_1-i\bar{\mu}_2)}{\Gamma(2w+2v)} \tag{5.3}$$

For ν real, the corresponding argument (including the integrals that arise from the (anti-) holomorphic discrete series) is even simpler (see [DG1] and [Zh2]). In this case, the asymptotic formula of $\mathcal{K}_\nu(s,v,w,\chi_\nu)$ becomes

$$\begin{aligned} \mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) &= B(v,w,\mu_1,\mu_2) \cdot \left(1 + |t+t_\nu|\right)^{-w} \\ &\quad \cdot \left[1 \;+\; \mathcal{O}_{\sigma,\, v,\, w,\, \mu_1,\, \mu_2}\left(\left(1+|t+t_\nu|\right)^{-1}\right)\right] \end{aligned} \tag{5.4}$$

where

$$B(v,w,\mu_1,\mu_2) \;=\; 2^{w-2}\, \pi^{-v}\, \frac{\Gamma(\frac{w+v+i\mu_1+i\mu_2}{2})\Gamma(\frac{w+v-i\mu_1+i\mu_2}{2})\Gamma(\frac{w+v+i\mu_1-i\mu_2}{2})\Gamma(\frac{w+v-i\mu_1-i\mu_2}{2})}{\Gamma(w+v)}$$

It now follows that for $\Re(w)$ sufficiently large,

$$I(0,w) \;=\; \sum_{\chi\in\widehat{C}_{0,S}} \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} L\big(\tfrac{1}{2}-it,,\; f_1\otimes\overline{\chi}\big)\cdot L\big(\tfrac{1}{2}+it,,\; \bar{f}_2\otimes\chi\big)\,\mathcal{K}_\infty(\tfrac{1}{2}+it,\,0,\,w,\,\chi)\,dt \tag{5.5}$$

Since $I(0,w)$ has analytic continuation to $\Re(w) > 11/18$, a mean value result can already be established by standard arguments. For instance, assume $f_1 = f_2 = \bar{f}$, and choose a function $h(w)$ which is holomorphic and with sufficient decay (in $|\Im(w)|$) in a suitable vertical strip containing $\Re(w) = 1$. For example, one can choose a suitable product of gamma functions. Consider the integral

$$\frac{1}{i}\int\limits_{\Re(w)=L} I(0,w)\,h(w)\,T^w\,dw \tag{5.6}$$

with L a large positive constant. Assuming $h(1) = 1$, we have the asymptotic formula

$$\sum_{\chi\in\widehat{C}_{0,S}}\int\limits_{-\infty}^{\infty} |L(\tfrac{1}{2}+it,f\otimes\chi)|^2\cdot M_{\chi,\,T}(t)\,dt \;\sim\; A\,T\,(\log T)^{r_1+r_2} \tag{5.7}$$

for some computable positive constant A, where

$$M_{\chi,\,T}(t) \;=\; \frac{1}{2\pi i}\int\limits_{\Re(w)=L} \mathcal{K}_\infty(\tfrac{1}{2}+it,\,0,\,w,\,\chi)\,h(w)\,T^w\,dw \tag{5.8}$$

For a character $\chi\in\widehat{C}_0$, put

$$\kappa_\chi(t) \;=\; \prod_{\nu\approx\mathbb{R}}(1+|t+t_\nu|)\;\cdot\;\prod_{\nu\approx\mathbb{C}}\big(1+\ell_\nu^2+4(t+t_\nu)^2\big) \qquad (t\in\mathbb{R}) \tag{5.9}$$

where it_ν and ℓ_ν are the parameters of the local component χ_ν of χ. Since χ is trivial on the positive reals,

$$\sum_{\nu|\infty} d_\nu\,t_\nu = 0$$

with $d_\nu = [k_\nu : \mathbb{R}]$ the local degree. For applications, it might be more convenient to work with a slightly modified function $Z(w)$ defined by

$$Z(w) \;=\; \sum_{\chi\in\widehat{C}_{0,S}}\int\limits_{-\infty}^{\infty} |L(\tfrac{1}{2}+it,f\otimes\chi)|^2\cdot\kappa_\chi(t)^{-w}\,dt \tag{5.10}$$

obtained from the function $I(0,w)$ by taking just the main terms in the asymptotic formulas (5.2) and (5.4) of the local components $\mathcal{K}_\nu(s,0,w,\chi_\nu)$. Its analytic properties can be transferred (with some technical adjustments) from those of $I(0,w)$. As an illustration of this fact, we show that the right-hand side of (5.10) is absolutely convergent for $\Re(w) > 1$. Using the asymptotic formulae

(5.2) and (5.4), it clearly suffices to verify the absolute convergence of the right-hand side of (5.5), with $f_1 = f_2 = \bar{f}$, when $w > 1$.

To see the absolute convergence of the defining expression (5.5) for $I(0, w)$, first note that the triple integral expressing $\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu)$ can be written as

$$\mathcal{K}_\nu(\tfrac{1}{2}+it,\, 0,\, w,\, \chi_\nu) \;=\; (2\pi)^3 \int_0^{\frac{\pi}{2}} (\cos\phi)^{2w-1} \sin\phi \cdot |V_{\mu_{f,\nu},\,\chi_\nu}(t,\phi)|^2 \, d\phi \qquad (\text{for } \nu \approx \mathbb{C}) \tag{5.11}$$

when $v = 0$ and $\Re(s) = \frac{1}{2}$, where

$$V_{\mu_{f,\nu},\,\chi_\nu}(t,\phi) \;=\; \int_0^\infty u^{2i(t_\nu+t)} K_{2i\mu_{f,\nu}}(4\pi u\cos\phi) J_{|\ell_\nu|}(4\pi u \sin\phi)\, du \tag{5.12}$$

Here we also used the well-known identity $J_{-\ell_\nu}(z) = (-1)^{\ell_\nu} J_{\ell_\nu}(z)$. The convergence of the last integral is justified by 6.576, integral 3, page 716 in [GR]. For $\nu \approx \mathbb{R}$, the local integral (3.9) has a similar form, when $v = 0$ and $\Re(s) = 1/2$, as it can be easily verified by a straightforward computation.

The form of the integral (5.11) allows us to adopt the argument used in the proof of Landau's Lemma to our context giving the desired conclusion. We shall follow [C], proof of Theorem 6, page 115.

Choose a sufficiently large real number a such that the right-hand side of (5.5) is convergent at $w = a$. Since $I(0, w)$ is holomorphic for $\Re(w) > 1$, its Taylor series

$$\begin{aligned} &\sum_{j=0}^\infty \frac{(w-a)^j}{j!}\, I^{(j)}(0,a) \\ &= \frac{1}{2\pi} \sum_{j=0}^\infty \frac{(w-a)^j}{j!} \sum_{\chi\in\widehat{C}_{0,S}} \int_{-\infty}^\infty |L(\tfrac{1}{2}+it, f\otimes\chi)|^2 \cdot \mathcal{K}_\infty^{(j)}(\tfrac{1}{2}+it, 0, a, \chi)\, dt \end{aligned} \tag{5.13}$$

has radius of convergence $a - 1$. Using the structure of (5.11) and its analog at real places, we have that

$$(w-a)^j \cdot \mathcal{K}_\infty^{(j)}(\tfrac{1}{2}+it, 0, a, \chi) \;\geq\; 0 \qquad (\text{for } w \leq a)$$

Having all terms non-negative in (5.13) when $w < a$, we can interchange the first sum with the second and the integral. Since

$$\mathcal{K}_\infty(\tfrac{1}{2}+it, 0, w, \chi) \;=\; \sum_{j=0}^\infty \frac{(w-a)^j}{j!}\, \mathcal{K}_\infty^{(j)}(\tfrac{1}{2}+it, 0, a, \chi)$$

the absolute convergence of (5.10) for $\Re(w) > 1$ follows.

Setting $w = 1 + \varepsilon$, then for arbitrary $T > 1$,

$$\sum_{\chi\in\widehat{C}_{0,S}} \int_{\mathfrak{I}_\chi(T)} |L(\tfrac{1}{2}+it, f\otimes\chi)|^2 \cdot T^{-1-\varepsilon}\, dt \;<\; Z(1+\varepsilon) \;\ll_\varepsilon\; 1$$

where $\mathfrak{I}_\chi(T) = \{t \in \mathbb{R} : \kappa_\chi(t) \le T\}$, and hence

$$\sum_{\chi \in \widehat{C}_{0,S}} \int_{\mathfrak{I}_\chi(T)} |L(\tfrac{1}{2}+it, f \otimes \chi)|^2 \, dt \ll_\varepsilon T^{1+\varepsilon} \tag{5.14}$$

Only finitely many characters contribute to the left-hand sum. This estimate is *compatible* with the convexity bound, in the sense that it implies for example that

$$\int_0^T |L(\tfrac{1}{2}+it, f)|^2 \, dt \ll_\varepsilon T^{[k:\mathbb{Q}]+\varepsilon}$$

Therefore, the function $Z(w)$ defined by (5.10) leads to averages of *reasonable* size suitable for applications. We return to a further study of the analytic properties of this function in a forthcoming paper.

Concluding remarks: The choice (3.11) of the data φ_ν at archimedean places was made for simplicity, to illustrate the non-vacuousness of the structural framework. Specifically, this choice yields cogent asymptotics, and gives an averaging that is *not too long*, i.e., is compatible with the convexity bound. This choice sufficed for our purposes, which were to stress generality, leaving aside more technical issues necessary to obtain sharper results. Its use allowed quick understanding of the size of the averages via the pole at $w = 1$.

The function $I(v, w)$ in (5.1) is analytic for v in a neighborhood of 0 and $\Re(w)$ sufficiently large, from the analytic properties of $\mathcal{K}_\infty(s, v, w, \chi)$ in Section 5. By computing $I(v, w)$ using (4.13), this observation can be used to find the value of the constant κ given in (4.7).

§Appendix 1. Convergence of Poincaré series

The aim of this appendix is to discuss the proofs of Proposition 2.6 and Theorem 2.7. Given the lack of complete arguments in the literature, we have given a full account, applicable more generally. For a careful discussion of some aspects of $GL(2)$, see [GJ] and [CPS1]. Note that the latter source needs some small corrections in the inequalities on pages 28 and 29.

We first prove the absolute convergence of the Poincaré series, uniformly on compacts on $G_\mathbb{A}$, for $G = GL_2$ over a number field k with ring of integers $\mathfrak{o}$, for $\Re(v) > 1$ and $\Re(w) > 1$. Second, we recall the notion of *norm* on a group, to prove convergence in L^2 for *admissible* data (see the end of Section 2), also reproving pointwise convergence by a more broadly applicable method.

Toward our first goal, we need an elementary comparison of sums and integrals under mild hypotheses. Let $V_1, \ldots, V_n$ be finite-dimensional real vector spaces, with fixed inner products, and put

$$V = V_1 \oplus \ldots \oplus V_n \qquad \text{(orthogonal direct sum)}$$

with the natural inner product. Fix a lattice Λ in V, and let F be a period parallelogram for Λ in V, containing 0. Let g be a real-valued function on V with $g(\xi) \ge 1$, such that $1/g$ has finite *integral* over V, and is *multiplicatively bounded* on each translate $\xi + F$, in the sense that, for each $\xi \in \Lambda$,

$$\sup_{y \in \xi+F} \frac{1}{g(y)} \ll \inf_{y \in \xi+F} \frac{1}{g(y)} \qquad \text{(with implied constant independent of } \xi)$$

For a differentiable function f, let $\nabla_i f$ be the gradient of f in the V_i variable. Then,

$$\sum_{\xi\in\Lambda} |f(\xi)| \ll \int_V |f(\xi)|\, d\xi \;+\; \sum_i \sup_{\xi\in V} \Big(g(\xi)\cdot|\nabla_i f(\xi)|\Big)$$

with the implied constant independent of f.

The following calculus argument gives this comparison (Abel summation). Let $\mathrm{vol}(\Lambda)$ be the natural measure of V/Λ. Certainly,

$$\mathrm{vol}(\Lambda)\cdot\sum_{\xi\in\Lambda}|f(\xi)| \;=\; \sum_{\xi\in\Lambda}|f(\xi)|\cdot\int_{\xi+F} dx$$

and

$$f(\xi)\int_{\xi+F} dx \;=\; \int_{\xi+F}(f(\xi)-f(x))\,dx \;+\; \int_{\xi+F} f(x)\,dx$$

The sum over $\xi\in\Lambda$ of the latter integrals is obviously the integral of f over V, as in the claim. The differences $f(\xi)-f(x)$ require further work. For $i=1,\ldots,n$, let x_i and y_i be the V_i–components of $x,y\in V$, respectively. Let

$$d_i(F) \;=\; \sup_{x,y\in F}|x_i-y_i|$$

By the Mean Value Theorem, we have the easy estimate

$$|f(\xi)-f(x)| \;\le\; \sum_{i=1}^n d_i(F)\cdot\sup_{y\in\xi+F}|\nabla_i f(y)|$$

Then,

$$\sum_{\xi\in\Lambda}\int_{\xi+F}|f(\xi)-f(x)|\,dx \;\ll\; \sum_{\xi\in\Lambda}\sum_{i=1}^n\sup_{y\in\xi+F}|\nabla_i f(y)|$$

$$=\;\sum_{i=1}^n\sum_{\xi\in\Lambda}\sup_{y\in\xi+F}\left(\frac{1}{g(y)}\,g(y)|\nabla_i f(y)|\right) \;\le\; \sum_{i=1}^n\sum_{\xi\in\Lambda}\Big(\sup_{y\in\xi+F}\frac{1}{g(y)}\Big)\cdot\Big(\sup_{y\in V} g(y)\,|\nabla_i f(y)|\Big)$$

$$\ll\; \int_V\frac{du}{g(u)}\cdot\sum_i\sup_{y\in V}\big(g(y)|\nabla_i f(y)|\big) \;\ll\; \sum_i\sup_{y\in V}\big(g(y)|\nabla_i f(y)|\big)$$

This gives the indicated estimate.

The above estimate will show that the Poincaré series with parameter v is dominated by the sum of an Eisenstein series at v and an Eisenstein series at $v+1+\varepsilon$ for every $\varepsilon>0$, under mild assumptions on the archimedean data. Such an Eisenstein series converges absolutely and uniformly on compacts for $\Re(v)>1$, either by Godement's criterion, in classical guise in [B], or by more elementary estimates that suffice for GL_2. Thus, the Poincaré series converges absolutely and uniformly for $\Re(v)>1$.

The assumptions on the archimedean data

$$\Phi_\infty(x) \;=\; \varphi_\infty\begin{pmatrix}1 & x\\ & 1\end{pmatrix}$$

are that

$$\int_{k_\infty} |\Phi_\infty(\xi)|\, d\xi \;<\; +\infty$$

and, letting ∇_ν be the gradient along the summand k_ν of k_∞, that, for each $\nu|\infty$,

$$\sup_{\xi\in k_\infty} |\nabla_\nu \Phi_\infty(\xi)| \;<\; \infty$$

The comparison argument is as follows. To make a vector from which to form an Eisenstein series, left-average the kernel

$$\varphi\left(\begin{pmatrix} a & \\ & d\end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right) \;=\; |a/d|^v \cdot \Phi(x) \qquad\qquad \text{(extended by right } K_{\mathbb{A}}\text{–invariance)}$$

for the Poincaré series over N_k. That is, form

$$\widetilde{\varphi}(g) \;=\; \sum_{\beta\in N_k} \varphi(\beta\cdot g)$$

This must be proven to be dominated by a vector (or vectors) from which Eisenstein series are formed. The usual vector for standard spherical Eisenstein series is

$$\eta_s\begin{pmatrix} a & * \\ & d\end{pmatrix} \;=\; |a/d|^s$$

extended to $G_{\mathbb{A}}$ by right $K_{\mathbb{A}}$–invariance. We claim that

$$\widetilde{\varphi} \;\ll\; \eta_v + \eta_{v+1+\varepsilon} \qquad\qquad \text{(for all } \varepsilon > 0)$$

Since all functions φ, $\widetilde{\varphi}$ and η_s are right $K_{\mathbb{A}}$–invariant and have trivial central character, it suffices to consider $g = nh$ with $n \in N_{\mathbb{A}}$ and

$$h \;=\; \begin{pmatrix} y & \\ & 1\end{pmatrix} \;\in\; H_{\mathbb{A}}$$

Let

$$n_t \;=\; \begin{pmatrix} 1 & t \\ & 1\end{pmatrix}$$

We have

$$\varphi(n_\xi\cdot n_x h) \;=\; \varphi(h\cdot h^{-1} n_\xi n_x h) \;=\; \varphi(h\cdot h^{-1} n_{\xi+x} h) \;=\; |y|^v\cdot \Phi\left(\tfrac{1}{y}\cdot(\xi+x)\right)$$

Thus, to dominate the Poincaré series by an Eisenstein series, it suffices to prove that

$$\sum_{\xi\in k} \Phi\left(\tfrac{1}{y}\cdot(\xi+x)\right) \;\ll\; 1 \;+\; |y| \qquad\qquad \text{(uniformly in } x\in N_{\mathbb{A}},\ y\in\mathbb{J})$$

Since $\widetilde{\varphi}$ is left N_k–invariant, it suffices to take $x \in \mathbb{A}$ to lie in a set of representatives X for $\mathbb{A}/k$, such as

$$X \;=\; k_\infty/\mathfrak{o} \;\oplus\; \textstyle\prod_{\nu<\infty} \mathfrak{o}_\nu$$

where, by abuse of notation, $k_\infty/\mathfrak{o}$ refers to a period parallelogram for the lattice $\mathfrak{o}$ in k_∞. As φ and $\widetilde{\varphi}$ are left H_k–invariant, so we can adjust y in $\mathbb{J}$ by $k^\times$. Since $\mathbb{J}^1/k^\times$ is compact, we can choose representatives in $\mathbb{J}$ for $\mathbb{J}/k^\times$ lying in $C' \cdot (0, +\infty)$ for some compact set $C' \subset \mathbb{J}^1$, with $(0, +\infty)$ embedded in $\mathbb{J}$ as usual by

$$t \longrightarrow (t^{1/n}, t^{1/n}, \dots, t^{1/n}, 1, 1, \dots, 1, \dots) \qquad \text{(non-trivial entries at archimedean places)}$$

where $n = [k : \mathbb{Q}]$. Further, for simplicity, we may adjust the representatives y such that $|y|_\nu \leq 1$ for all finite primes ν. The compactness of C' implies that

$$|y| \;\leq\; \textstyle\prod_{\nu|\infty} |y_\nu|_\nu \;\ll\; |y| \qquad \text{(with implied constant depending only on } k\text{)}$$

Likewise, due to the compactness, the archimedean valuations of representatives have bounded ratios.

At a finite place, $\Phi_\nu\left(\frac{1}{y} \cdot (x + \xi)\right)$ vanishes unless

$$\frac{1}{y} \cdot (x + \xi) \;\in\; \mathfrak{o}_\nu$$

That is, since we want a uniform bound in $x \in \mathfrak{o}_\nu$, this vanishes unless

$$\xi \;\in\; \mathfrak{o}_\nu + y \cdot \mathfrak{o}_\nu \subset \mathfrak{o}_\nu$$

since we have taken representatives y with y_ν integral at all finite ν. Thus, the sum over ξ in k reduces to a sum over ξ with archimedean part in the lattice $\Lambda = \mathfrak{o} \subset k_\infty$.

Setting up a comparison as above, let

$$V \;=\; k_\infty \;=\; \bigoplus_{\nu|\infty} k_\nu$$

Let $\mathrm{vol}(\Lambda)$ be the volume of Λ. For archimedean place ν let ∇_ν be the gradient along k_ν, and $d_\nu(\Lambda)$ the maximum of $|x_\nu - y_\nu|_\nu$ for $x, y \in F$, a fixed period parallelogram for Λ in k_∞. We have

$$\sum_{\xi \in \Lambda} \Phi\left(\tfrac{1}{y} \cdot (\xi + x)\right) \;\ll\; \int_{k_\infty} \Phi_\infty\left(\tfrac{1}{y} \cdot (\xi + x)\right) d\xi \;+\; \sum_{\nu|\infty} \sup_{\xi \in k_\infty} \Big(g(\xi) \cdot |\nabla_\nu \Phi_\infty(\xi)|\Big)$$

for any suitable weight function g. In the integral, replace ξ by $\xi - x$, and then by $\xi \cdot y$, to see that

$$\int_{k_\infty} \Phi_\infty\left(\tfrac{1}{y} \cdot (\xi + x)\right) d\xi \;=\; |y|_\infty \cdot \int_{k_\infty} \Phi_\infty(\xi)\, d\xi \;\ll\; |y| \cdot \int_{k_\infty} \Phi_\infty(\xi)\, d\xi$$

with the implied constant depending only upon k, using the choice of representatives y for $\mathbb{J}/k^\times$.

To estimate the sum, for $x \in k_\infty$, fix $\varepsilon > 0$ and take weight function

$$g(\xi) \;=\; \textstyle\prod_{\nu|\infty} (1 + |\xi|_\nu^2)^{\frac{1}{2}+\varepsilon}$$

This is readily checked to have the *multiplicative boundedness* property needed: the function g is continuous, and for $|\xi| \geq 2|x|$, we have the elementary

$$\tfrac{1}{2} \cdot |\xi| \;\leq\; |\xi - x| \;\leq\; 2 \cdot |\xi|$$

from which readily follows the bound for $g(\xi)$.

What remains is to compute the indicated supremums with attention to their dependence on y. At an archimedean place ν,

$$\sup_{\xi\in k_\infty} \left(g(\xi)\cdot|\nabla_\nu\Phi_\nu\left(\tfrac{1}{y}\cdot(\xi+x)\right)|\right) \;=\; \sup_{\xi\in k_\infty} \left(g(\xi-x)\cdot|\nabla_\nu\Phi_\nu\left(\tfrac{1}{y}\cdot\xi\right)|\right)$$
$$\ll\; \sup_{\xi\in k_\infty} \left(g(\xi)\cdot|\nabla_\nu\Phi_\nu\left(\tfrac{1}{y}\cdot\xi\right)|\right)$$

by using the boundedness property of g. Then replace ξ by $\xi\cdot y$, to obtain

$$\sup_{\xi\in k_\infty} \left(g(y\cdot\xi)\cdot|\nabla_\nu\Phi_\nu(\xi)|\right)$$

Since

$$(1+|y|_\nu^2|\xi|_\nu^2) \;\le\; (1+|y|_\nu^2)\cdot(1+|\xi|_\nu^2) \qquad\qquad \text{(for all } \nu|\infty)$$

we have $g(y\cdot\xi)\le g(y)\cdot g(\xi)$, and

$$\sup_{\xi\in k_\infty} \left(g(y\cdot\xi)\cdot|\nabla_\nu\Phi_\nu(\xi)|\right) \;\le\; g(y)\cdot \sup_{\xi\in k_\infty} \left(g(\xi)\cdot|\nabla_\nu\Phi_\nu(\xi)|\right)$$

Here the weighted supremums of the gradients appear, which we have assumed finite.

Finally, estimate

$$g(y) \;=\; \prod_{\nu|\infty}(1+|y|_\nu^2)$$

with y in our specially chosen set of representatives. For these representatives, for any two archimedean places ν_1 and ν_2, we have

$$|y|_{\nu_1}^{n_{\nu_1}} \;\ll\; |y|_{\nu_2}^{n_{\nu_2}}$$

where the n_{ν_i} are the local degrees $n_{\nu_i}=[k_{\nu_i}:\mathbb{R}]$. Therefore,

$$|y|_\nu \;\ll\; |y|^{n_\nu/n}$$

where $n=\sum_\nu n_\nu$ is the global degree. Thus,

$$\prod_{\nu|\infty}(1+|y|_\nu^2) \;\ll\; 1+|y|^2$$

Then,

$$\prod_{\nu|\infty}(1+|y|_\nu^2)^{\frac{1}{2}+\varepsilon} \;\ll\; (1+|y|^2)^{\frac{1}{2}+\varepsilon}$$

Putting this all together, for every $\varepsilon>0$

$$\widetilde{\varphi}\begin{pmatrix} y & * \\ 0 & 1\end{pmatrix} \;\ll\; |y|^v\cdot(1+|y|^2)^{\frac{1}{2}+\varepsilon} \;=\; |y|^v+|y|^{v+1+2\varepsilon} \;=\; \eta_v\begin{pmatrix} y & * \\ 0 & 1\end{pmatrix}+\eta_{v+1+2\varepsilon}\begin{pmatrix} y & * \\ 0 & 1\end{pmatrix}$$

which is the desired domination of the Poincaré series by a sum of Eisenstein series.

For the particular choice of archimedean data

$$\Phi_\infty(\xi) \;=\; \prod_{\nu\approx\mathbb{R}} \frac{1}{(1+\xi^2)^{w_\nu/2}} \cdot \prod_{\nu\approx\mathbb{C}} \frac{1}{(1+\bar{\xi}\,\xi)^{w_\nu}}$$

the *integrability* condition is met when $\Re(w_\nu) > 1$ for all archimedean ν. Similarly, the weighted supremums of gradients are *finite* for $\Re(w_\nu) > 1$.

Altogether, this particular Poincaré series is absolutely convergent for $\Re(v) > 1 + 2\varepsilon$ and $\Re(w_\nu) > 1 + \varepsilon$, for every $\varepsilon > 0$. This proves Proposition 2.6.

Soft convergence estimates on Poincaré series: Now we give a different approach to convergence, more convenient for proving square integrability of Poincaré series. It is more robust, and does also reprove pointwise convergence, but gives a weaker result than the previous more explicit approach. Let G be a (locally compact, Hausdorff, separable) unimodular topological group. Fix a compact subgroup K of G. A *norm* $g \longrightarrow \|g\|$ on G is a positive real-valued *continuous* function on G with properties

- $\|g\| \geq 1$ and $\|g^{-1}\| = \|g\|$
- *Submultiplicativity*: $\|gh\| \leq \|g\| \cdot \|h\|$
- *K–invariance*: for $g \in G$, $k \in K$, $\|k \cdot g\| = \|g \cdot k\| = \|g\|$
- *Integrability*: for some $\sigma_o \geq 0$

$$\int_G \|g\|^{-\sigma}\, dt \;<\; +\infty \qquad\qquad (\text{for all } \sigma > \sigma_o)$$

For a *discrete* subgroup Γ of G, we claim the corresponding *summability*:

$$\sum_{\gamma\in\Gamma} \frac{1}{\|\gamma\|^\sigma} \;<\; +\infty \qquad\qquad (\text{for all } \sigma > \sigma_o)$$

The proof is as follows. From

$$\|\gamma \cdot g\| \;\leq\; \|\gamma\| \cdot \|g\|$$

for $\sigma > 0$

$$\frac{1}{\|\gamma\|^\sigma \cdot \|g\|^\sigma} \;\leq\; \frac{1}{\|\gamma \cdot g\|^\sigma}$$

Invoking the discreteness of Γ in G, let C be a small open neighborhood of $1 \in G$ such that

$$C \cap \Gamma \;=\; \{1\}$$

Then,

$$\int_C \frac{dg}{\|g\|^\sigma} \cdot \sum_{\gamma\in\Gamma} \frac{1}{\|\gamma\|^\sigma} \;\leq\; \int_C \sum_{\gamma\in\Gamma} \frac{1}{\|\gamma \cdot g\|^\sigma}\, dg \;=\; \sum_{\gamma\in\Gamma} \int_{\gamma^{-1}C} \frac{1}{\|g\|^\sigma}\, dg \;\leq\; \int_G \frac{dg}{\|g\|^\sigma} \;<\; +\infty$$

This gives the indicated summability. Let H be a closed subgroup of G, and define a *relative* norm

$$\|g\|_{H} \;=\; \inf_{h\in H\cap\Gamma} \|h \cdot g\|$$

From the definition, there is the left $H \cap \Gamma$–invariance

$$\|h \cdot g\|_{H} = \|g\|_{H} \qquad \text{(for all } h \in H \cap \Gamma\text{)}$$

Note that $\| \ \|_{H}$ depends upon the discrete subgroup Γ.

Moderate increase, sufficient decay: Let H be a closed subgroup of G. A left $H \cap \Gamma$–invariant complex-valued function f on G is of *moderate growth modulo* $H \cap \Gamma$, when, for some $\sigma > 0$,

$$|f(g)| \ll \|g\|_{H}^{\sigma}$$

The function f is *rapidly decreasing modulo* $H \cap \Gamma$ if

$$|f(g)| \ll \|g\|_{H}^{-\sigma} \qquad \text{(for } \textit{all } \sigma > 0\text{)}$$

The function f is *sufficiently rapidly decreasing modulo* $H \cap \Gamma$ (for a given purpose) if

$$|f(g)| \ll \|g\|_{H}^{-\sigma} \qquad \text{(for } \textit{some} \text{ sufficiently large } \sigma > 0\text{)}$$

Since $\|g\|_{H}$ is an infimum, for $\sigma > 0$ the power $\|g\|^{-\sigma}$ is a supremum

$$\frac{1}{\|g\|_{H}^{\sigma}} = \sup_{h \in H \cap \Gamma} \frac{1}{\|hg\|^{\sigma}}$$

Pointwise convergence of Poincaré series: We claim that, for f left $H \cap \Gamma$–invariant and sufficiently rapidly decreasing mod $H \cap \Gamma$, the *Poincaré series*

$$P_f(g) = \sum_{\gamma \in (H \cap \Gamma)\backslash \Gamma} f(\gamma \cdot g)$$

converges absolutely and uniformly on compacts. To see this, first note that, for all $h \in H \cap \Gamma$,

$$\|\gamma\|_{H} \le \|h \cdot \gamma\| = \|h \cdot \gamma g \cdot g^{-1}\| \le \|h\gamma g\| \cdot \|g^{-1}\|$$

Thus, taking the inf over $h \in H \cap \Gamma$,

$$\frac{\|\gamma\|_{H}}{\|g^{-1}\|} \le \|\gamma \cdot g\|_{H}$$

Thus, for $\sigma > 0$,

$$\frac{1}{\|\gamma \cdot g\|_{H}^{\sigma}} \le \frac{\|g\|^{\sigma}}{\|\gamma\|_{H}^{\sigma}}$$

and

$$P_f(g) = \sum_{\gamma \in (H \cap \Gamma)\backslash \Gamma} f(\gamma \cdot g) \ll \sum_{\gamma \in (H \cap \Gamma)\backslash \Gamma} \frac{1}{\|\gamma \cdot g\|_{H}^{\sigma}} \le \|g\|^{\sigma} \cdot \sum_{\gamma \in (H \cap \Gamma)\backslash \Gamma} \frac{1}{\|\gamma\|_{H}^{\sigma}}$$

$$\le \|g\|^{\sigma} \cdot \sum_{\gamma \in (H \cap \Gamma)\backslash \Gamma} \sum_{h \in H \cap \Gamma} \frac{1}{\|h \cdot \gamma\|^{\sigma}} = \|g\|^{\sigma} \cdot \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|^{\sigma}} \ll \|g\|^{\sigma} \qquad \text{(for } \sigma > \sigma_o\text{)}$$

estimating a sup of positive terms by the sum, for $\sigma > \sigma_o$ to assure that the sum over Γ converges.

Moderate growth of Poincaré series: Next, we claim that Poincaré series are of moderate growth modulo Γ, namely, that

$$P_f(g) \ll \|g\|_\Gamma^\sigma \qquad \text{(for all } \sigma > \sigma_o\text{)}$$

Indeed, the previous estimate is uniform in g, and the left-hand side is Γ–invariant. That is, for all $\gamma \in \Gamma$,

$$P_f(g) = P_f(\gamma \cdot g) \ll \|\gamma \cdot g\|^\sigma \qquad \text{(for all } \sigma > \sigma_o\text{, with implied constant independent of } g, \gamma\text{)}$$

Taking the inf over γ gives the assertion.

Square integrability of Poincaré series: Next, we claim that for f left $H \cap \Gamma$–invariant and sufficiently rapidly decreasing mod $H \cap \Gamma$, P_f is *square-integrable* on $\Gamma\backslash G$. Specifically, assume that

$$|f(g)| \ll |g|^{-2\sigma} \qquad \text{(for some } \sigma > \sigma_o\text{)}$$

Unwind, and use the assumed estimate on f along with the above-proven moderate growth of the Poincaré series:

$$\int_{\Gamma\backslash G} |P_f|^2 = \int_{(H\cap\Gamma)\backslash G} |f| \cdot |P_f| \ll \int_{(H\cap\Gamma)\backslash G} \|g\|_H^{-2\sigma} \cdot \|g\|_H^\sigma \, dg$$

Estimating a sup by a sum, and unwinding further,

$$\int_{(H\cap\Gamma)\backslash G} \|g\|_H^{-\sigma} \, dg \le \int_{(H\cap\Gamma)\backslash G} \sum_{h\in(H\cap\Gamma)\backslash\Gamma} \|h \cdot g\|^{-\sigma} \, dg = \int_G \|g\|^{-\sigma} \, dg < +\infty$$

since $\sigma > \sigma_o$. This proves the square integrability of the Poincaré series.

Construction of a norm on $PGL_2(\mathbb{A})$: We want a norm on $G = PGL_2(\mathbb{A})$ over a number field k that meets the conditions above, including the integrability, with K the image in $PGL_2(\mathbb{A})$ of the maximal compact

$$\prod_{\nu\approx\mathbb{R}} O_2(\mathbb{R}) \times \prod_{\nu\approx\mathbb{C}} U(2) \times \prod_{\nu<\infty} GL_2(\mathfrak{o}_\nu)$$

of $GL_2(\mathbb{A})$. We take Γ to be the image in $PGL_2(\mathbb{A})$ of $GL_2(k)$. Let $\mathfrak{g}$ be the algebraic Lie algebra of GL_2 over k, so that, at each place ν of k,

$$\mathfrak{g}_\nu = \{\text{2-by-2 matrices with entries in } k_\nu\}$$

Let ρ denote the Adjoint representation of GL_2 on $\mathfrak{g}$, namely,

$$\rho(g)(x) = gxg^{-1} \qquad \text{(for } g \in GL_2 \text{ and } x \in \mathfrak{g}\text{)}$$

The kernel of ρ on GL_2 is the center Z, so the image G of GL_2 under ρ is PGL_2. As expected, let

$$G_\nu = \rho(GL_2(k_\nu)) = GL_2(k_\nu)/Z_\nu \qquad K_\nu = \rho(GL_2(o_\nu)) = GL_2(\mathfrak{o}_\nu)/(Z_\nu \cap GL_2(\mathfrak{o}_\nu))$$

and

$$\Gamma = G_k = \rho(GL_2(k)) = GL_2(k)/Z_k$$

Since Γ is a subgroup of $GL_k(\mathfrak{g}_k)$, it is discrete in the adelization of $GL_k(\mathfrak{g}_k)$, so is discrete in $G_{\mathbb{A}}$. Let $\{e_{ij}\}$ be the 2-by-2 matrices with non-zero entry just at the $(i,j)^{\text{th}}$ location, where the entry is 1.

At an archimedean place ν of k, put a Hilbert space structure on $\mathfrak{g}_\nu$ by

$$\langle x, y\rangle = \mathrm{tr}(y^* x)$$

where y^* is y–transpose for ν real, and y–transpose-conjugate for ν complex. We put the usual (sup-norm) operator norm on linear operators T on $\mathfrak{g}_\nu$, namely

$$|T|_{\mathrm{op}} = \sup_{|x|\le 1} |Tx|$$

By design, since the inner product on $\mathfrak{g}_\nu$ is $\rho(K_\nu)$–invariant, this operator norm is invariant under $\rho(K_\nu)$. Note that at complex places this set-up effectively uses a classical normalization of the absolute value on $\mathbb{C}$, rather than the product-formula normalization.

For a non-archimedean local field k with norm $|\cdot|_\nu$ and ring of integers $\mathfrak{o}$, give $\mathfrak{g}_\nu$ the sup-norm

$$|\textstyle\sum_{ij} a_{ij}\, e_{ij}| = \sup_{ij} |a_{ij}|_\nu \qquad\qquad (\text{with } a_{ij} \in k_\nu)$$

There is the *operator norm* on $GL_{k_\nu}(\mathfrak{g}_\nu)$ given by

$$|g|_{\mathrm{op}} = \sup_{x\in V,\ |x|\le 1} |g\cdot x|$$

By design, this norm is invariant under $\rho(K_\nu)$.

Norms on local groups and adele groups: For any place ν of k, define a (local) *norm* $\|g\|_\nu$ on the image $G_\nu = PGL_2(k_\nu)$ of $GL_2(k_\nu)$ in $GL_{k_\nu}(\mathfrak{g}_\nu)$ by

$$\|g\|_\nu = \max\{|g|_{\mathrm{op}},\ |g^{-1}|_{\mathrm{op}}\}$$

Since the norm on $\mathfrak{g}_\nu$ is K_ν–invariant, and K_ν is stable under inverse, the operator norms are left and right K_ν–invariant, and the norms $\|\ \|_\nu$ are left and right K_ν–invariant. Note that for $\nu < \infty$ the operator norm is 1 on K_ν. To prove that

$$\|g\cdot h\|_\nu \le \|g\|_\nu \cdot \|h\|_\nu$$

use the definition:

$$\|g\cdot h\|_\nu = \max\{|gh|_{\mathrm{op}},\ |h^{-1}g^{-1}|_{\mathrm{op}}\}$$

$$\le \max\{|g|_{\mathrm{op}}\cdot|h|_{\mathrm{op}},\ |g^{-1}|_{\mathrm{op}}\cdot|h^{-1}|_{\mathrm{op}}\} \le \max\{|g|_{\mathrm{op}},\ |g^{-1}|_{\mathrm{op}}\}\cdot\max\{|h|_{\mathrm{op}},\ |h^{-1}|_{\mathrm{op}}\} = \|g\|_\nu\cdot\|h\|_\nu$$

For $g = \{g_\nu\}$ in the adele group $G_{\mathbb{A}}$, let

$$\|g\| = \textstyle\prod_\nu \|g_\nu\|_\nu$$

The factors in the product are 1 for all but finitely many places. The left and right K–invariance for $K = \prod_\nu K_\nu$ follows from the local K_ν–invariance. Invariance under inverse is likewise clear.

Integrability: Toward integrability, we explicitly bound the local integrals

$$\int_{G_\nu} \|g\|_\nu^{-\sigma}\, dg$$

At finite primes, use the p–adic Cartan decomposition (here just the elementary divisor theorem) inherited from $GL_2(k_\nu)$ via the quotient map, namely,

$$G_\nu \;=\; \bigsqcup_{\delta \in A_\nu/(A_\nu \cap Z_\nu K_\nu)} K_\nu \cdot \delta \cdot K_\nu \qquad \text{(where } A_\nu \text{ is diagonal matrices)}$$

By conjugating by permutation matrices and adjusting by Z_ν, we may assume, further, that

$$\delta \;=\; \begin{pmatrix} \delta_1 & \\ & 1 \end{pmatrix} \qquad \text{(with } |\delta_1| \geq 1)$$

For any choice ϖ_ν of local parameter for k_ν, we may adjust by $A_\nu \cap K_\nu$ so that δ_1 is a power of ϖ. On a given K_ν double coset, the norm is

$$\|K_\nu \cdot \delta \cdot K_\nu\|_\nu \;=\; \|\delta\|_\nu \;=\; \max\{|\rho(\delta)|_{\rm op},\; |\rho(\delta^{-1})|_{\rm op}\} \;=\; \max\{|\delta_1|_\nu,\; |\delta_1|_\nu^{-1}\}$$

and

$$\mathrm{meas}\,(K_\nu \delta K_\nu) \;=\; \mathrm{meas}\,(K_\nu) \cdot \mathrm{card}(K_\nu\backslash K_\nu \delta K_\nu)$$

Let $q = q_\nu$ be the residue field cardinality, and let $|\delta|_\nu = q^\ell$ with $\ell \geq 0$. Then,

$$\mathrm{card}\; K_\nu\backslash K_\nu \delta K_\nu \;=\; \mathrm{card}\;(K_\nu \cap \delta^{-1} K_\nu \delta)\backslash K_\nu \;\leq\; \mathrm{card}\; K_\nu(\ell)\backslash K_\nu$$

where $K_\nu(\ell)$ is a sort of congruence subgroup, namely,

$$K_\nu(\ell) \;=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_\nu \;:\; c \in \varpi^\ell \cdot \mathfrak{o}_\nu \right\}$$

Let $K'_\nu = \{g \in K_\nu : g = I_2 \mod \varpi\mathfrak{o}\}$, and let $\mathbb{F}_q$ be the finite field with q elements. We have an elementary estimate

$$[K_\nu : K_\nu(1)] \;=\; \frac{[K_\nu : K'_\nu]}{[K_\nu(1) : K'_\nu]} \;=\; \mathrm{card}\{\text{lines in } \mathbb{F}_q^2\} \;=\; \frac{q^2-1}{q-1} \;=\; q+1$$

and

$$[K_\nu(\ell) : K_\nu(\ell+1)] \;=\; q \qquad \text{(for } \ell \geq 1)$$

Thus,

$$[K_\nu : K_\nu(\ell)] \;\leq\; (1+\frac{1}{q}) \cdot q^\ell \qquad \text{(for } \ell \geq 1)$$

Thus, the integral of $\|g\|_\nu^{-\sigma}$ has an upper bound

$$\int_{G_\nu} \frac{dg}{\|g\|_\nu^\sigma} \;\leq\; 1 + (1+\frac{1}{q}) \sum_{\ell \geq 1} q^{-\sigma\ell} \cdot q^\ell \;\leq\; 1 + (1+\frac{1}{q}) \sum_{\ell \geq 1} (q^{1-\sigma})^\ell$$

For $\sigma > 1$, the geometric series converges. Thus,

$$\int_{G_\nu} \frac{dg}{\|g\|_\nu^\sigma} \;\le\; 1 + (1+\frac{1}{q}) \cdot \frac{q^{1-\sigma}}{1-q^{1-\sigma}} \;=\; \frac{1+q^{-\sigma}}{1-q^{1-\sigma}} \;=\; \frac{1-q^{-2\sigma}}{(1-q^{-\sigma})(1-q^{1-\sigma})}$$

Note that there is *no* leading constant.

The integrability condition on the adele group can be verified by showing the finiteness of the product of the corresponding local integrals. Since there are only finitely many archimedean places, it suffices to consider the product over finite places. By comparison to the zeta function of the number field k,

$$\prod_{\nu<\infty} \frac{1-q_\nu^{-2\sigma}}{(1-q_\nu^{-\sigma})(1-q_\nu^{1-\sigma})} \;<\; \infty \qquad\qquad (\text{for } \sigma > 2)$$

Thus, letting G_{fin} be the finite-prime part of the idele group $G_{\mathbb{A}}$,

$$\int_{G_{\mathrm{fin}}} \frac{dg}{\|g\|^\sigma} \;=\; \prod_{\nu<\infty} \int_{G_\nu} \frac{dg}{\|g\|_\nu^\sigma} \;<\; +\infty \qquad\qquad (\text{for } \sigma > 2)$$

For integrability locally at archimedean places, exploit the left and right K_ν–invariance, via Weyl's integration formula. Let A_ν be the image under Ad of the standard maximal *split* torus from $GL_2(k_\nu)$, namely, real diagonal matrices. Let $\Phi^+ = \{\alpha\}$ be the singleton set of standard positive roots of A_ν, namely

$$\alpha \;:\; \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \longrightarrow a_1/a_2$$

$\mathfrak{g}_\alpha$ be the α–rootspace, and, for $a \in A_\nu$, let

$$D(a) \;=\; |\alpha(a) - \alpha^{-1}(a)|^{\dim_{\mathbb{R}} \mathfrak{g}_\alpha}$$

with the classical absolute value on $\mathbb{C}$ (*not* the product formula normalization). The Weyl formula for a left and right K_ν–invariant function f on G_ν is

$$\int_{G_\nu} f(g)\, dg \;=\; \int_{A_\nu} D(a) \cdot f(a)\, da$$

with Haar measure on A_ν. For PGL_2, the dimension $\dim_{\mathbb{R}} \mathfrak{g}_\alpha$ is 1 for $k_\nu \approx \mathbb{R}$ and is 2 for $k_\nu \approx \mathbb{C}$. The norm of a diagonal element is easily computed via the adjoint action on $\mathfrak{g}_\nu$, namely

$$\|a\|_\nu \;=\; \max\{|a_1/a_2|,\, |a_2/a_1|\}$$

with the classical absolute values on $\mathbb{R}$ or $\mathbb{C}$. Thus,

$$D(a) \;\ll\; \|a\|_\nu^{d_\nu} \qquad\qquad (\text{with } d_\nu = [k_\nu : \mathbb{R}])$$

Thus, the integral over $PGL_2(k_\nu)$ is dominated by a one-dimensional integral, namely,

$$\int_{G_\nu} \frac{dg}{\|g\|_\nu^\sigma} \;=\; \int_{A_\nu} \frac{D(a)}{\|a\|_\nu^\sigma}\, da \;\ll\; \int_0^\infty (\max(|t|, |t|^{-1})^{d_\nu - \sigma} \frac{dt}{t} \qquad\qquad (\text{with } d_\nu = [k_\nu : \mathbb{R}])$$

The latter integral is estimated by

$$\int_{k_\nu^\times} (\max(|x|,|x|^{-1})^{-\beta}\,dx \;=\; \int_{|x|\le 1}(|x|^{-1})^{-\beta}\,dx + \int_{|x|\ge 1}|x|^{-\beta}\,dx \;<\; +\infty \qquad \text{(for } \beta>1)$$

for ν either real or complex. Note that the norm is that occurring in the product formula. This gives the desired local integrability for $\sigma > 2$ at archimedean places, and completes the proof of global integrability. That is, we can take $\sigma_o = 2$.

Poincaré series for GL_2: Recall the context of Sections 2 and 3. Let $G = GL_2(\mathbb{A})$ over a number field k, Z the center of GL_2, and K_ν the standard maximal compact in G_ν. Let

$$M \;=\; \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\} \qquad N \;=\; \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$$

To form a Poincaré series, let $\varphi = \bigotimes_\nu \varphi_\nu$, where each φ_ν is right K_ν–invariant, Z_ν–invariant, and on G_ν

$$\varphi_\nu\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \;=\; |a|_\nu^v \cdot \Phi_\nu(x)$$

where at finite primes Φ_ν is the characteristic function of the local integers $\mathfrak{o}_\nu$. At archimedean places, we assume that Φ_ν is sufficiently continuously differentiable, and that these derivatives are absolutely integrable. The global function φ is left M_k–invariant, by the product formula. Then, let

$$f(g) \;=\; \int_{N_{\mathbb{A}}} \overline{\psi}(n)\,\varphi(ng)\,dn$$

where ψ is a standard non-trivial character on $N_k\backslash N_{\mathbb{A}} \approx k\backslash\mathbb{A}$. As in (4.6), but with slightly different notation, the Poincaré series of interest is

$$\text{Pé}^*(g) \quad = \sum_{\gamma\in Z_k N_k\backslash G_k} f(\gamma\cdot g)$$

Convergence uniformly pointwise and in L^2: From above, to show that this converges absolutely and uniformly on compacts, and also that it is in $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}})$, use a norm on the group $PGL_2 = GL_2/Z$, take $\Gamma = PGL_2(k)$, and show that f is *sufficiently rapidly decreasing on $PGL_2(\mathbb{A})$ modulo N_k*.

To give the sufficient decay modulo N_k, it suffices to prove sufficient decay of $f(nm)$ for n in a well-chosen set of representatives for $N_k\backslash N_{\mathbb{A}}$, and for m among representatives

$$m \;=\; \begin{pmatrix} a & \\ & 1 \end{pmatrix}$$

for $M_{\mathbb{A}}/Z_{\mathbb{A}}$. For $m \in M_{\mathbb{A}}$ and $n \in N_{\mathbb{A}}$, the submultiplicativity $\|nm\| \le \|n\|\cdot\|m\|$ gives

$$\frac{1}{\|n\|^\sigma\cdot\|m\|^\sigma} \;\le\; \frac{1}{\|nm\|^\sigma} \qquad \text{(for } \sigma>0)$$

That is, roughly put, it suffices to prove decay in $N_{\mathbb{A}}$ and $M_{\mathbb{A}}$ separately. Since $N_k \backslash N_{\mathbb{A}}$ has a set of representatives E that is *compact*, on such a set of representatives the norm is *bounded*. Thus, it suffices to prove that

$$f(nm) \ll \frac{1}{\|m\|^\sigma} \qquad \left(\text{for } n \in E, \text{ and } m = \begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)$$

Since f factors over primes, as does $\|m\|$, it suffices to give suitable *local* estimates.

At finite ν, the ν^{th} local factor of f is left ψ–equivariant by N_ν, and

$$f_\nu(nm) \;=\; \psi(n) \cdot \int_{N_\nu} \overline{\psi}(n')\, \varphi_\nu(n'm)\, dn' \;=\; \psi(n) \cdot \int_{N_\nu} \overline{\psi}(n')\, \varphi_\nu(m \cdot m^{-1} n' m)\, dn'$$

$$= \; \psi(n) \cdot |a|_\nu \cdot \int_{k_\nu} \overline{\psi}_o(ax)\, |a|_\nu^v \cdot \Phi_\nu(x)\, dx$$

where for finite ν Φ_ν is the characteristic function of the local integers, and

$$\psi \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \;=\; \overline{\psi}_o(x) \qquad\qquad m \;=\; \begin{pmatrix} a & \\ & 1 \end{pmatrix}$$

Thus,

$$|f_\nu(nm)| \;=\; |a|_\nu^{\Re(v)+1} \cdot \int_{k_\nu} \overline{\psi}_o(ax)\, \Phi_\nu(x)\, dx \;=\; |a|_\nu^{\Re(v)+1} \cdot \widehat{\Phi}_\nu(a)$$

At every finite place ν, Φ_ν has compact support, and at almost every finite ν, $\widehat{\Phi}_\nu$ is simply the characteristic function of $\mathfrak{o}_\nu$. Thus, almost everywhere,

$$|f_\nu(nm)| \;\le\; |a|_\nu^{\Re(v)+1} \cdot \widehat{\Phi}_\nu(a) \;\le\; \big(\max\{|a|_\nu,\, |a|_\nu^{-1}\}\big)^{-(\Re(v)+1)} \;=\; \|m\|_\nu^{-(\Re(v)+1)}$$

At the finitely many finite places where $\widehat{\Phi}$ is not exactly the characteristic function of $\mathfrak{o}_\nu$, the same argument still gives the weaker but sufficient estimate

$$|f_\nu(nm)| \;\le\; |a|_\nu^{\Re(v)+1} \cdot \widehat{\Phi}_\nu(a) \;\ll\; \big(\max\{|a|_\nu,\, |a|_\nu^{-1}\}\big)^{-(\Re(v)+1)} \;=\; \|m\|_\nu^{-(\Re(v)+1)}$$

Thus, we have the finite-prime estimate

$$\prod_{\nu<\infty} |f_\nu(nm)| \;\ll\; \prod_{\nu<\infty} \|m\|_\nu^{-(\Re(v)+1)}$$

From above, the sufficient decay condition for square integrability is

$$\Re(v) + 1 \;>\; 2\sigma_0 \;=\; 4$$

so we need $\Re(v) > 3$.

At archimedean places, given $\ell > 0$, for Φ_ν sufficiently differentiable with absolutely integrable derivatives, ordinary Fourier transform theory implies that

$$|\widehat{\Phi}_\nu(a)| \;\ll\; (1 + |a|_\nu)^{-\ell}$$

Thus, from the general local calculation above,

$$|f_\nu(nm)| \;=\; |a|_\nu^{\Re(v)+1} \cdot \widehat{\Phi}_\nu(a) \;\ll\; |a|_\nu^{\Re(v)+1} \cdot (1+|a|_\nu)^{-\ell} \;\ll\; \begin{cases} \|m\|_\nu^{-(\Re(v)+1))} & (\text{for } |a|_\nu \leq 1) \\ \|m\|_\nu^{-(\ell-\Re(v)-1)} & (\text{for } |a|_\nu \geq 1) \end{cases}$$

Thus, for sufficient decay for square-integrability, we need

$$\Re(v) + 1 \;>\; 2\sigma_0 \;=\; 4$$

or $\Re(v) > 3$, the same condition as from the finite-prime discussion. Further, we need

$$\ell - \Re(v) - 1 \;>\; 2\sigma_0 \;=\; 4$$

which is $\ell > \Re(v) + 5$.

In summary, for $\Re(v) > 3$ and $\ell > \Re(v) + 5$, the function f has sufficient decay so that the associated Poincaré series $\text{Pé}^* = P_f$ converges uniformly on compacts, and is in $L^2(Z_{\mathbb{A}}GL_2(k)\backslash GL_2(\mathbb{A}))$. This proves Theorem 2.7.

§**Appendix 2. Mellin transform of Eisenstein Whittaker functions**

The computation discussed in this appendix was needed in the proof of Proposition 4.10. While the details of this computation are given below, we also cite [W2], Chapter VII, for standard facts about the Tate-Iwasawa theory of zeta integrals.

The global Mellin transform of W^E factors

$$\int_{\mathbb{J}} |a|^v \, W^E_{s,\chi} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da \;=\; \prod_\nu \int_{k_\nu^\times} |a|_\nu^v \, W^E_{s,\chi,\nu} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da$$

To compute this, we cannot simply change the order of integration, since this would produce a divergent integral along the way. Instead, we present the vectors η_ν in a different form. Let Φ_ν be any Schwartz function on k_ν^2 invariant under K_ν (under the obvious right action of GL_2), and put

$$\eta'_\nu(g) = \chi_\nu(\det g)\,|\det g|_\nu^s \cdot \int_{k_\nu^\times} \chi_\nu^2(t)\,|t|_\nu^{2s} \cdot \Phi_\nu(t \cdot e_2 \cdot g)\,dt$$

where $e_2 = e_{2,\nu}$ is the second basis element in k_ν^2. This η'_ν has the same left P_ν–equivariance as η_ν, namely

$$\eta'_\nu\left(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \cdot g\right) \;=\; |a/d|_\nu^s \cdot \chi_\nu(a/d) \cdot \eta'_\nu(g)$$

For Φ_ν invariant under the standard maximal compact K_ν of $GL_2(k_\nu)$, the function η'_ν is right K_ν–invariant. By the Iwasawa decomposition, up to constant multiples, there is only one such function, so

$$\eta'_\nu(g) = \eta'_\nu(1) \cdot \eta_\nu(g) \qquad\qquad (\text{since } \eta_\nu(1) = 1)$$

and[1]

$$\eta'_\nu(1) = \int_{k_\nu^\times} \chi^2(t)\,|t|^{2s} \cdot \Phi(t \cdot e_2 \cdot 1)\,dt = \zeta_\nu(2s,\,\chi^2,\,\Phi(0,*)) \qquad (\text{a Tate-Iwasawa zeta integral})$$

[1]From now on, to avoid clutter, suppress the subscript ν where there is no risk of confusion. For instance, we shall write $|\cdot|$, ψ, χ, etc., rather than $|\cdot|_\nu$, ψ_ν, χ_ν, etc.

Thus, it suffices to compute the local Mellin transform of

$$\eta'_\nu(1)\cdot W^E_{s,\chi,\nu}(m) = \int_{N_\nu} \overline{\psi}(n)\,\eta'_\nu(w_\circ nm)\,dn = \chi(a)|a|^s \int_{N_\nu} \overline{\psi}(n) \int_{k_\nu^\times} \chi^2(t)|t|^{2s}\,\Phi(t\cdot e_2\cdot w_\circ nm)\,dt\,dn$$

$$= \chi(a)|a|^s \int_{k_\nu} \overline{\psi}(x) \int_{k_\nu^\times} \chi^2(t)|t|^{2s}\,\Phi(tx,ta)\,dt\,dx \qquad \left(\text{with } m = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

At *finite* primes ν, we may as well take Φ to be

$$\Phi(t,x) = \mathrm{ch}_{\mathfrak{o}_\nu}(t)\cdot\mathrm{ch}_{\mathfrak{o}_\nu}(x) \qquad (\mathrm{ch}_X = \text{characteristic function of set } X)$$

Then $\eta'_\nu(1)$ is exactly an L–factor (see [W2], page 119, Proposition 10)

$$\eta'_\nu(1) = \zeta_\nu(2s,\,\chi^2,\,\mathrm{ch}_{\mathfrak{o}_\nu}) = L_\nu(2s,\,\chi^2)$$

and (for further details see [W2], page 107, Corollary 1, and page 108, Corollary 3),

$$\eta'_\nu(1)\cdot W^E_{s,\chi,\nu}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \chi(a)|a|^s \int_{k_\nu} \overline{\psi}(x)\,\mathrm{ch}_{\mathfrak{o}_\nu}(tx) \int_{k_\nu^\times} \chi^2(t)|t|^{2s}\,\mathrm{ch}_{\mathfrak{o}_\nu}(ta)\,dt\,dx$$

$$= \chi(a)|a|^s\,\mathrm{meas}\,(\mathfrak{o}_\nu) \int_{k_\nu^\times} \mathrm{ch}_{\mathfrak{o}_\nu^*}(1/t)\chi^2(t)|t|^{2s-1}\,\mathrm{ch}_{\mathfrak{o}_\nu}(ta)\,dt$$

$$= |\mathfrak{d}_\nu|^{1/2}\cdot\chi(a)|a|^s \int_{k_\nu^\times} \mathrm{ch}_{\mathfrak{o}_\nu^*}(1/t)\chi^2(t)|t|^{2s-1}\,\mathrm{ch}_{\mathfrak{o}_\nu}(ta)\,dt$$

where $\mathfrak{d}_\nu \in k_\nu^\times$ is such that $(\mathfrak{o}_\nu^*)^{-1} = \mathfrak{d}_\nu\cdot\mathfrak{o}_\nu$. We can compute now the Mellin transform

$$\int_{k_\nu^\times} |a|^v\cdot\left(\chi(a)|a|^s \int_{k_\nu^\times} \mathrm{ch}_{\mathfrak{o}_\nu^*}(1/t)\chi^2(t)|t|^{2s-1}\,\mathrm{ch}_{\mathfrak{o}_\nu}(ta)\,dt\right)da$$

Replace a by a/t, and then t by $1/t$ to obtain a product of two zeta integrals

$$\left(\int_{k_\nu^\times} |a|^v\cdot\chi(a)|a|^s\,\mathrm{ch}_{\mathfrak{o}_\nu}(a)\,da\right)\cdot\left(\int_{k_\nu^\times} \mathrm{ch}_{\mathfrak{o}_\nu^*}(1/t)\chi(t)|t|^{s-1-v}\,dt\right)$$

$$= \zeta_\nu(v+s,\,\chi,\,\mathrm{ch}_{\mathfrak{o}_\nu})\cdot\zeta_\nu(v+1-s,\,\overline{\chi},\,\mathrm{ch}_{\mathfrak{o}_\nu^{*-1}})$$

$$= L_\nu(v+s,\,\chi)\cdot L_\nu(v+1-s,\,\overline{\chi})\cdot|\mathfrak{d}_\nu|^{-(v+1-s)}\,\chi(\mathfrak{d}_\nu)$$

Thus, dividing through by $\eta'_\nu(1)$ and putting back the measure constant, the Mellin transform of $W^E_{s,\chi,\nu}$ is

$$\int_{k_\nu^\times} |a|^v\,W^E_{s,\chi,\nu}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da = |\mathfrak{d}_\nu|^{1/2}\cdot\frac{L_\nu(v+s,\,\chi)\cdot L_\nu(v+1-s,\overline{\chi})}{L_\nu(2s,\,\chi^2)}\cdot|\mathfrak{d}_\nu|^{-(v+1-s)}\,\chi(\mathfrak{d}_\nu)$$

Let $\mathfrak{d}$ be the idele whose ν^{th} component is $\mathfrak{d}_\nu$ for finite ν and whose archimedean components are all 1. The product over all finite primes ν of these local factors is

$$\int_{\mathbb{J}^{\text{fin}}} |a|^v\,W^E_{s,\chi}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da = |\mathfrak{d}|^{1/2}\cdot\frac{L(v+s,\,\chi)\cdot L(v+1-s,\,\overline{\chi})}{L(2s,\,\chi^2)}\cdot|\mathfrak{d}|^{-(v+1-s)}\,\chi(\mathfrak{d})$$

In our application, we will replace s by $1-s$ and χ by $\overline{\chi}$, giving

$$\int_{\mathbb{J}^{\mathrm{fin}}} |a|^v \, W^E_{1-s,\,\overline{\chi}} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da = |\mathfrak{d}|^{1/2} \cdot \frac{L(v+1-s,\,\overline{\chi}) \cdot L(v+s,\,\chi)}{L(2-2s,\,\overline{\chi}^2)} \cdot |\mathfrak{d}|^{-(v+s)}\,\overline{\chi}(\mathfrak{d})$$

In particular, with χ trivial,

$$\int_{\mathbb{J}^{\mathrm{fin}}} |a|^v \, W^E_{1-s} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da = |\mathfrak{d}|^{1/2} \cdot \frac{\zeta_k(v+1-s) \cdot \zeta_k(v+s)}{\zeta_k(2-2s)} \cdot |\mathfrak{d}|^{-(v+s)}$$

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